University of Rochester
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PHYSICS 401 (MATHEMATICAL METHODS I)
Professor Emil Wolf

Class notes taken by Jaime E. Villate
SOME USEFUL BOOKS (GENERAL)


H. B. Jeffreys: Methods of Mathematical Physics, (Cambridge University Press)


H. W. Wyld: Mathematical Methods for Physics, (Benjamin, Reading, Mass.).
SOME BOOKS ON ANALYSIS AND FUNCTIONS OF A COMPLEX VARIABLE


W. Kaplan: *Advanced Calculus*, (Cambridge University Press).


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Lecture 1, 9/4/85

Elementary Analysis in the Complex Domain.

\[ z = x + iy \quad , \quad x, y \text{ real} \]
\[ z = r e^{i\theta} \quad r = \text{modulus of } z \text{ (amplitude of } |z|) \]
\[ \theta = \text{arg } z \quad (\text{phase of } z) \]

Principle value of \( \theta = (-\pi, \pi) \)

\[ |z_1 + z_2| \leq |z_1| + |z_2| \]
\[ |z_1 - z_2| \geq | |z_1| - |z_2| | \quad (\text{Copson, } \S 132, \text{ p7}) \]

in general,
\[ |\sum z_i| \leq \sum |z_i| \]

Function of a complex variable.

For real functions we start with a set \( D \text{ (domain)} \).
\[ W = f(z) \]

\[ \text{Domain (set of points)} \]

\[ \text{Complex plane} \]
\[ W(z) = w(x+iy) = u(x,y) + i v(x,y) \]

**Example:**
\[ f(x) = \frac{1}{1 + x^2} \]

\[ \frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \ldots \], if \(|x| < 1\)

It is not easy to see why the series diverges for \(|x| \geq 1\).

But it is easily shown, if we define the complex function:

\[ f(z) = \frac{1}{1 + z^2} \]

\[ f(z) \text{ is not defined at } z = \pm i \]
The point at infinity.

\[ z = \frac{1}{z} \]

By definition, the point at infinity is that \( z \) which corresponds to \( z' = 0 \), by the relation above.

\[ f(z) = f\left(\frac{1}{z'}\right) = g(z') \]

Then, the value of \( f(z) \) at infinity is the value of \( g(z') \) at \( z' = 0 \). (or \( \lim_{z \to 0} g(z') \))

Example. \( f(z) = \frac{z^2}{z^2 + z^2} \) (\( a = \text{const.} \))

\[ g(z') = \frac{1}{z'^2 + 1} \quad \lim_{z' \to 0} g(z') = 1 \]

RIEMANN SPHERE.

Another way of defining the point at infinity is due to Riemann (see Copson § 1.4).

Complex plane = Argand plane + point at infinity.
SETS AND SEQUENCES IN THE COMPLEX PLANE

Set of points: a set of points selected according to a certain rule.

Neighborhood of a point \( z_0 \) in the Argand plane.
\[
|z - z_0| < \varepsilon
\]

Neighborhood of the point at infinity.
\[
|z^2| < \varepsilon \quad \text{where} \quad z = \frac{1}{z^2}
\]
\[
\Rightarrow |\frac{1}{z^2}| < \varepsilon \quad |z| > \frac{1}{\varepsilon}
\]

Limit (or limiting) point of a set.
\( \xi \) is said to be a limiting point of a set \( S \), if every neighborhood of it contains at least one point of \( S \), distinct of \( \xi \). The limiting point needs not to be a member of the set.

Example.
\[
S = \{ \frac{1}{n} \mid n = 1, 2, \ldots \}
\]
the limit is \( z = 0 \), and it does not belong to \( S \).
**BOUNDDED SET.**

\[ \exists K > 0, \ |z| < K \text{ for every } z \text{ in the set.} \]

**Example 1.**

\[ \{z_n\}, \ z_n = \frac{i}{n} \quad (n = 1, 2, 3, \ldots) \]

\[ |z_n| \leq 1 \]

Then, \( \{z_n\} \) is bounded

**Example 2.**

\[ \{z_n\}, \ z_n = i^n \quad (n = 1, 2, 3, \ldots) \]

is unbounded.

**THEOREM (Bolzano and Weierstrass)**

Every bounded infinite set has at least one limit.

There is an infinite number of points inside the square. Then one of the two rectangles must have infinite points. Take the one which has infinite points and so on.
ENUMERABLE AND NON-ENUMERABLE SETS.

A set is enumerable if there is a one-to-one correspondence between the points of it and the natural numbers.

Examples.

1. \(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\)  Enumerable

2. All real numbers in the interval \((0,1)\)  
   Not enumerable

This is one of the theorems due to Cantor which allow to classify the infinities.

SEQUENCE OF POINTS.

An enumerable set with a certain order

\[\{z_n\} \quad z_1, z_2, z_3, \ldots \neq z_3, z_1, z_2, \ldots\]

CONVERGENT INFINITE SEQUENCE.

\(\{z_n\}\) is said to be convergent if it has one and only one limiting point \(z\) and \(z\) is not the point at infinity.
\[ z_n \to \xi \text{ as } n \to \infty \]

This definition implies that every neighborhood of \( \xi \) will contain all the points of \( z_n \), except a finite number of them.

**LIMIT OF A SEQUENCE.**

\( \xi \) is the limit of \( \{z_n\} \), if,

\[ \exists \varepsilon > 0; \ \forall \delta > 0 \quad |z_n - \xi| < \delta \quad \forall n \geq n_0(\delta) \]

---

**Lecture 2, Sept. 9/85**

**DIVERGENT SEQUENCE.**

Any infinite sequence which is not convergent.

**Examples.**

1. \( \{z_n\} = \frac{1}{n} \quad \text{(n=1,2,3,...)} \)
   is a convergent sequence. Its limit is \( \xi = 0 \)

2. \( \{z_n\} = \frac{(-1)^{n-1}}{n} \quad \text{(n=1,2,3,...)} \)
   converges to \( \xi = 0 \)

3. \( \{z_n\} = (-1)^n \left(1 + \frac{1}{n}\right) \quad \text{(n=1,2,3,...)} \)
Is a divergent sequence

CAUCHY’S PRINCIPLE OF CONVERGENCE

\( \{a_n\} \) is convergent if \( \varepsilon > 0 \); \( \exists N(\varepsilon) \) such that \( |a_n - a_m| < \varepsilon \)

(and only if)

for \( p = 1, 2, 3, \ldots \)

INFINITE SERIES

Given a sequence \( \{a_n\} \) we can define a sequence of partial sums.

\[ s_1 = a_1 \]
\[ s_2 = a_1 + a_2 \]
\[ \vdots \]
\[ s_n = a_1 + a_2 + \cdots + a_n \]

If the sequence \( \{s_n\} \) is convergent, its limit(s) is called the sum of the series.

\[ s = \lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} a_n \]

If the sequence \( \{s_n\} \) is divergent, we say that the series diverges.
CAUCHY'S PRINCIPLE OF CONVERGENCE

\[ \varepsilon > 0; \exists N(\varepsilon) \text{ s.t. } |S_{N+p} - S_N| \leq \varepsilon \quad \forall p = 2, 3, \ldots \]

\[ \Rightarrow |a_{N+1} + \cdots + a_{N+p}| \leq \sum_{n=0}^{\infty} a_n \text{ converge.} \]

\[ p = 1 : \quad |a_{N+1}| \leq \varepsilon \Rightarrow a_n \to 0 \text{ as } n \to \infty \]

But this condition is not sufficient,

Example: \( a_n = \frac{1}{n} \to 0 \text{ as } n \to \infty \)

but \( \sum_{n=0}^{\infty} a_n \) diverges

Operations which do not affect convergence:

1. Insertion of parenthesis around a finite number of terms

2. Addition and subtraction term by term

3. Rearrangement of the order of a finite number of terms. (Knopp pp. 75-76)

ABSOLUTE AND CONDITIONAL CONVERGENCE

\[ \sum_{n=0}^{\infty} a_n \text{ is absolutely convergent if } \sum_{n=0}^{\infty} |a_n| \text{ is convergent.} \]
A convergent series which is not absolutely convergent is called conditionally convergent.

**Theorem.**

If a series is absolutely convergent, it is convergent.

**Proof:**

\[ S_N = \sum_{n=1}^{N} a_n \]

\[ S_N = \sum_{n=1}^{N} |a_n| \]

\[ \exists \varepsilon > 0; \exists N \ni |S_{N+P} - S_N| < \varepsilon, \forall p = 1, 2, 3, \ldots \]

\[ |a_{N+1}| + |a_{N+2}| + \cdots + |a_{N+p}| < \varepsilon \]

\[ |S_{N+P} - S_N| = |a_{N+1} + \cdots + a_{N+p}| \leq |S_{N+P} - S_N| \]

\[ \Rightarrow |S_{N+P} - S_N| < \varepsilon \]

The opposite is not true:

**Example**

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \text{ Converges} \]

\[ 1 + \frac{1}{2} + \frac{1}{3} \text{ not.} \]
THEOREM

If a series is absolutely convergent, then its sum is not affected by any change on the order of its terms.

Suppose the conditional convergent series
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \to \log 2 \]
\[ 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} \to \frac{3}{2} \log 2 \]
\[ 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} + \frac{1}{7} + \ldots \to \frac{1}{2} \log 6 \]

THEOREM

If a conditionally convergent series consist of real numbers only, it can be arranged so that it converges to any number.

MULTIPLICATION OF SERIES.

Let \( \sum_{n=1}^{\infty} a_n = A \) \( \sum_{n=1}^{\infty} b_n = B \) be absol. conv.

\( (AB) = \left( \sum_{n=1}^{\infty} a_n \right) \left( \sum_{n=1}^{\infty} b_n \right) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n b_m \)

then the series \( \sum_{k=1}^{\infty} p_k \) containing all the terms \( p_k = a_nb_m \) in any order, converges absolutely to \( AB \)
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TESTS FOR ABSOLUTE CONVERGENCE
(Sufficiency criteria)

1. d'Alembert or ratio test.

\[
\text{if } \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \text{ exists, (in which case we call)}
\]

then \[ \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent if } \delta < 1 \text{ and divergent if } \delta \geq 1 \]

Example

\[ S = \sum_{n=1}^{\infty} \frac{z^n}{n^\mu} \quad \mu \text{ arbitrary} \]

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{z}{(n+1)\mu} n^\mu \right| = \left| \frac{z}{(n+1)\mu} \right| \geq 1
\]

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |z|
\]

\[ \left\{ \begin{array}{l} |z| < 1, \text{ S is absolutely convergent} \\ |z| \geq 1, \text{ S diverges.} \end{array} \right. \]
(2) CAUCHY’S NTH ROOT TEST

This test is convenient for power series.

If \( \lim_{n \to \infty} \sqrt[n]{|a_n|} \) exists, (8), then

\[ \sum_{n=1}^{\infty} |a_n| \text{ is absolutely convergent if } s < 1, \]

and divergent if \( s > 1 \)

Example. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2)^n} \quad (z \neq 0) \]

\[ \sqrt[n]{|a_n|} = \sqrt[n]{\left| \frac{(-1)^n}{(n^2)^n} \right|} = \frac{1}{\sqrt[n]{n^{2n}}} = \frac{1}{n^{\frac{2}{n}}} \]

\[ \Rightarrow s = \lim_{n \to \infty} \sqrt[n]{|a_n|} = 0 \]

Therefore, the series converges absolutely.

(3) RAABE’S TEST.

It is particularly useful when \( s = 1 \) is obtained in the ratio test.

If \( \lim_{n \to \infty} \left\{ n \left| \frac{a_{n+1}}{a_n} \right| - 1 \right\} \) exists, (8)

then if \( s < 1 \), \( \sum_{n=1}^{\infty} a_n \) converges absolutely.
Example: \( \sum_{n=1}^{\infty} \frac{z^n}{1+n^2} \), for \( z \) on the unit circle \( |z|=1 \)

\[
\left| \frac{2n+1}{2n} \right| = \left| \frac{z^{n+1}}{z^n} \right| \frac{1+n^2}{1+(n+1)^2} = |z| \frac{1+n^2}{1+(n+1)^2} = \frac{1+n^2}{1+(n+1)^2}
\]

\[ s = \lim_{n \to \infty} \left| \frac{2n+1}{2n} \right| = 1 \]

\[
N \left( \left| \frac{2n+1}{2n} \right| - 1 \right) = \frac{(2n-1)n}{1+(n+1)^2} \to -2 \text{ as } n \to \infty
\]

Therefore, the series converges absolutely, which is amazing, since this series is analytic inside \( |z|=1 \), but diverges outside.

4. COMPARISON TEST

\[
\sum_{n=1}^{\infty} a_n \text{, if there is a } \sum_{n=1}^{\infty} r_n \text{ converging positive series, then, if } N \text{, } |a_n| < r_n \text{ for } n \geq N
\]

\[
\Rightarrow \sum_{n=1}^{\infty} a_n \text{ is absolute convergent}
\]

Example: \( \sum_{n=1}^{\infty} \frac{n z^{n-1}}{(1+\frac{1}{n})^n - z^n} \), for \( |z| < 1 \)

\[
\left| (1+\frac{1}{n})^n - z^n \right| \geq \left| (1+\frac{1}{n})^n - |z|^n \right| \geq 1 \quad \forall \quad |z| < 1
\]
\[ (1 + \frac{1}{n})^n - 1z^n > (1 + \frac{1}{n})^n - 1 \]
\[ \downarrow \quad 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2}\left(\frac{1}{n}\right)^2 \]
\[ \Rightarrow \left| \frac{n z^{n-1}}{(1 + \frac{1}{n})^n - z^n} \right| \leq n|z|^{n-1} = r_n, \text{ real and positive} \]

\[ \frac{r_{n+1}}{r_n} = \frac{n+1}{n} \cdot \frac{1}{|z|}, \quad \rightarrow \frac{1}{|z|} \quad \text{as} \quad n \to \infty \]

Therefore, \( \sum_{n=1}^{\infty} r_n \) converges and the series converges absolutely for \( |z| < 1 \).

**Continuous Functions.**

\[ f(z) \text{ is close to } l \quad (|f(z) - l| < \varepsilon) \quad \varepsilon > 0 \]
\[ z \text{ is close to } z_0 \quad (|z - z_0| < \delta) \]

\[ l = \lim_{z \to z_0} f(z) \text{ if } \varepsilon > 0; \exists \delta(\varepsilon, z_0), |f(z) - l| < \varepsilon \]
\[ 0 < |z - z_0| < \delta \]

The neighborhood of \( z_0 \), \( 0 < |z - z_0| < \delta \) is called **deleted neighborhood**.
\( f(z) \) is continuous at \( z = z_0 \) if:

1. \( f(z) \to l \) as \( z \to z_0 \)
2. \( l = f(z_0) \)

i.e. \( f(z) \to f(z_0) \) as \( z \to z_0 \)

**Definition.** A function \( f(z) \) defined in a domain \( D \) is said to be continuous at \( z = z_0 \), if,

\[ \exists \varepsilon > 0; \exists \delta(\varepsilon, z_0) \cdot |f(z) - f(z_0)| < \varepsilon \quad \text{and} \quad |z - z_0| < \delta \]

**Differentiability.**

If \( f(z) \) is one-valued at \( D \), \( f(z) \) is differentiable at \( z = z_0 \in D \) if

\[ \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \]

exists, provided \( z \in D \).

\[ \exists l \cdot \varepsilon > 0; \exists \delta(\varepsilon, z_0) \cdot \left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \varepsilon \quad \text{with} \quad 0 < |z - z_0| < \delta \]

\[ l = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \] = derivative of \( f(z) \)

at \( z = z_0 \).

We say that \( f(z) \) is differentiable in \( D \) if \( f(z) \) exists at every point in \( D \).
Continuity is a necessary condition for differentiability, but it is not sufficient.

Example: \(f(z) = |z|^2\) is continuous at every finite point of the complex plane.

\[
\frac{f(z)-f(z_0)}{z-z_0} = \frac{|z|^2 - |z_0|^2}{z-z_0} = \frac{zz^*-z_0z_0^*}{z-z_0} = \frac{z^*(z-z_0) + z_0(z^*-z_0^*)}{z-z_0} = z^* + z_0 \frac{z^*-z_0^*}{z-z_0}
\]

\[
\begin{align*}
z - z_0 &= \text{re}^{i\theta} \\
z^* - z_0^* &= \text{re}^{-i\theta} \\
\Rightarrow \frac{f(z)-f(z_0)}{z-z_0} &= z^* + z_0 (\cos 2\theta - i \sin \theta)
\end{align*}
\]

This is not a fixed number, but it depends on \(\theta\), unless \(z_0 = 0\).

Therefore the function \(f(z) = |z|^2\) is differentiable only at the origin, even though it is continuous everywhere.
Lecture 4, Sept. 16/85.

**ANALYTIC FUNCTIONS.**

If \( f(z) \) has derivative at every point, except a finite number, of a domain \( D \), we say that \( f(z) \) is analytic in \( D \). (Riemann 1851)

**Definition.**

\( f(z) \), one valued and differentiable at every point of \( D \) (except possibly at a finite number of points in \( D \)), is said to be analytic in \( D \).

The exception points are called SINGULARITIES of \( f(z) \) in the domain \( D \).

If \( f(z) \) has no singularities in \( D \) it is said to be REGULAR in \( D \). (or holomorphic)

**Theorem.** \( f(z), g(z) \) both regular in \( D \),

\[ \Rightarrow \]

i) \( f(z) \pm g(z) \) is regular in \( D \)

ii) \( f(z) \cdot g(z) \)

iii) \( \frac{f(z)}{g(z)} \) is analytic in \( D \) if \( g(z)=0 \) only in a finite number of points in \( D \).
**Theorem.** \( W \) is a regular analytic function of \( \xi \) in \( D \), and \( \xi = \xi(z) \) is regular in \( D \), then \( W(z) \) is regular in \( D \).

Also:
\[
\frac{dw}{dz} = \frac{dw}{d\xi} \frac{d\xi}{dz}
\]

**NECESSARY CONDITIONS FOR REGULARITY (Cauchy-Riemann equations)**

\[
f(z) = u(x,y) + iv(x,y)
\]

where \( z = x + iy \)

and \( u(x,y) , v(x,y) \) are real functions.

If \( \frac{df}{dz} \) exists, the limit \( \frac{f(z + \delta z) - f(z)}{\delta z} \) must be the same along paths I and II.

\[ I : \left( u(x+\delta x,y) + iv(x+\delta x,y) - u(x,y) - iv(x,y) \right) \delta x \]

\[ \rightarrow \frac{2u}{2x} + i \frac{2v}{2x} \]
\[ u(x, y+\delta y) + iv(x, y+\delta y) - u(x, y) - iv(x, y) \delta y \]
\[
\rightarrow \frac{\partial u}{\partial y} - i \frac{\partial v}{\partial y}
\]

EQUATING,

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{align*}
\]

Cauchy-Riemann differential equations

This equation was derived by D'Alembert in 1752 but not in a general sense.

**Theorem.**

If \( f(z) = u(x, y) + iv(x, y) \) then, it is necessary that \( u_x, u_y, v_x, v_y \) exist and satisfy Cauchy-Riemann equations.

**SUFFICIENT CONDITIONS FOR REGULARITY.**

(For a proof, see Copson pp 41-42).

**Theorem.** A continuous one-valued function \( f(z) = u(x, y) + iv(x, y) \) will be regular at \( z = x + iy \) if at that point \( u_x, u_y, v_x, v_y \) exist, are continuous, and satisfy Cauchy-Riemann equations.
**Definition**: **CONJUGATE FUNCTION**

\[ f(z) = u(x, y) + i v(x, y); \quad f^* = u - iv \]

Conseq. of C-R eqn.

1. \[ W = u(x, y) + iv(x, y) = W(z, z^*) \]
   \[ x = \frac{1}{2} (z + z^*) \]
   \[ y = \frac{1}{2i} (z - z^*) \]

**Examples**: 1. \[ W = |z|^2 = zz^* \]
   \[ \Rightarrow W \text{ is not analytic.} \]

2. \[ \cos(x + 2iy) = \cos(x + iy + iy) \]
   \[ = \cos(z + \frac{1}{2} (z - z^*)) \]
   \[ = \cos\left(\frac{3}{2}z - z^*\right) \]
   \[ \Rightarrow \cos(x + 2iy) \text{ is not analytic} \]

2. A function which is regular throughout a domain \( D \) can not be a real function, unless it is a constant function.

3. \[ u(x, y) = \text{const.} \quad v(x, y) = \text{const} \]

\[ u = \alpha \perp v = \beta \quad \text{if } f(z) \neq 0 \]
HARMONIC FUNCTIONS.

So far, we have seen that \( u \) and \( v \) must satisfy certain relations between them; but is it possible to choose any \( u \) and find \( v \)? The answer is no, because \( u \) and \( v \) must satisfy also certain condition independently.

\[
U_x = U_y \quad U_y = -U_x
\]

Assume second order derivatives exist. Then,

\[
\begin{align*}
U_{xx} &= U_{yx} \\
U_{yy} &= -U_{xy}
\end{align*}
\]

Since \( U_{yx} = U_{xy} \), (if \( U_{yx}, U_{xy} \) are continuous).

\[
U_{xx} + U_{yy} = 0
\]

In the same way,

\[
U_{xx} + U_{yy} = 0
\]

Such functions are called HARMONIC.

POWER SERIES.

\[ f(z) = A_0 + A_1 (z-a) + A_2 (z-a)^2 + \ldots \]

represents an analytic function in the domain where the series converges.
Theorem: if \( \sum_{n=0}^{\infty} A_n z^n \) converges (absolutely) for \( z_0 \), then it also converges absolutely for all \( z : |z| < |z_0| \).

Also, if \( \sum_{n=0}^{\infty} A_n z^n \) diverges at \( z_0 \), then it also diverges for any \( z : |z| > |z_0| \).

Therefore, for any power series there is always a circle in which the series converges and diverges outside it. The radius of that circle is called RADIUS OF CONVERGENCE.

CONVERGENCE OF POWER SERIES.

1. Ratio test:

\[
\left| \frac{A_{n+1} z^{n+1}}{A_n z^n} \right| = \left| \frac{A_{n+1}}{A_n} \right| |z| \rightarrow |z| \lim_{n \to \infty} \left| \frac{A_{n+1}}{A_n} \right|
\]

if \( \lim_{n \to \infty} \left| \frac{A_{n+1}}{A_n} \right| \) exists and is \( \frac{1}{R} \), then the series converges for \( |z| < R \).

Lemma: \( \phi(n) \to k \neq 0 \), as \( n \to \infty \); \( \frac{1}{\phi(n)} \to \frac{1}{k} \), as \( n \to \infty \).

Thus,

\[
R = \lim_{n \to \infty} \left| \frac{A_n}{A_{n+1}} \right| \quad (\text{if } \frac{1}{R} \neq 0)
\]

if \( \frac{1}{R} = \lim_{n \to \infty} \left| \frac{A_{n+1}}{A_n} \right| = 0 \), the series converges for every \( z \).
2. Cauchy's $n$th root test
\[
\sqrt[n]{|A^n|} = |z| \sqrt[n]{|A|^n} \rightarrow |z| \lim_{n \to \infty} \sqrt[n]{|A_n|}
\]
if \( \lim_{n \to \infty} \sqrt[n]{|A_n|} = \frac{1}{R} \), then the series converges for \(|z| < R\).

**Theorem.**

- Inside its circle of convergence, a power series represents a regular analytic function. (Copson pp. 38-39)

- The derivative of this function at any point inside the circle of convergence is
\[
f'(z) = \sum_{n=1}^{\infty} nA_n z^{n-1}
\]

Also, the radius of convergence of \( \sum_{n=1}^{\infty} nA_n z^{n-1} \) is the same as that of \( \sum_{n=1}^{\infty} A_n z^n \).

**Elementary Functions.**

- **Polynomial**
\[
P(z) = A_0 + A_1 z + \cdots + A_n z^n
\]
Therefore \( P(z) \) is regular for every finite point.
\( e^z \) is an entire function.

@ **Trigonometric functions.**

\[
\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots
\]

\[
\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots
\]

It is easy to show that both of them are entire.

\[
\tan z = \frac{\sin z}{\cos z}
\]

the only zeros of \( \cos z \) are those on the real axis (Copson pp 45-46).

\[
\sin^2 z + \cos^2 z = 1
\]

© **Hyperbolic functions.**

\[
\sinh z = \frac{1}{2}(e^z - e^{-z}) \quad \text{entire}
\]

\[
\cosh z = \frac{1}{2}(e^z + e^{-z})
\]

\[
\tanh z = \frac{\sinh z}{\cosh z}
\]

\[
\sinh iz = i \sin z \quad \cos iz = \cosh z
\]

\[
\sinh iz = i \sin z \quad \cosh iz = i \cos z
\]
Logarithmic function.

When $x$ is real and positive, $y = \log x$ is defined by $x = e^y$.

For a complex $z$,

$W = \log z$ is defined by $e^W = z$.

But, $e^{W+2\pi i} = e^W$.

Thus, $W = \log z$ is a many-valued function.

$z = |z| e^{i \arg(z)}$

$W = u + iv \quad u, v \in \mathbb{R}$

$\Rightarrow e^{u+iv} = |z| e^{i \arg z} = e^u e^{i\phi}$

$u = \log |z|$

$v = \arg z$

$\boxed{\log z = \log |z| + i \arg z}$

Principle value of $\log z$. $\rightarrow$ principle value of $\arg z$:

$-\pi < \arg z \leq \pi$. 
There is a discontinuity in the imaginary part of \( \log(z) \).

In any domain which does not include the negative part of the \( x \)-axis, \( \log z \) is regular and analytic.

**Theorem.** The principal value of \( \log z \) is a regular analytic function in any closed domain which does not include any point on the negative real axis, nor 0, and its derivative is \( \frac{1}{z} \).

\( \log z \) does not have a series expansion around the origin; however:

\[
\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \quad (R=1)
\]
For $x$ real and positive, and $\alpha$ is real,

$$x^\alpha = e^{\alpha \log x}$$

For $z$ and $\alpha$ complex, but $z \neq 0$,

$$z^\alpha = e^{\alpha \log z} \quad \text{(multivalued)}$$

Its principal value is analytic and regular in any domain which does not include the origin or any point on the negative real axis.

**Contour Integrals.**

Contours:

- Piecewise smooth

$$I_C = \int_C f(z) \, dz$$

$$\sum_{i=1}^{n} f(z_i) \Delta z_i \quad \text{where} \quad \Delta z_i = z_i - z_{i-1}$$

If, $\max_i |\Delta z_i| \to 0$ as $n \to \infty$

then, $\sum_{i=1}^{n} f(z_i) \Delta z_i \to \int_C f(z) \, dz$
\[ z = z(t) = x(t) + iy(t) \]

Examples:

\[ t = s = \text{arc length} \]

\[ t = x, \quad y = at + b \]

\[
\int_C f(z) \, dz = \int_C [u + iv][dx + idy] = \int_{t_0}^{t_1} [u(t) + iv(t)][\frac{dx}{dt} + i\frac{dy}{dt}] \, dt \\
= \int_{t_0}^{t_1} [u \frac{dx}{dt} - v \frac{dy}{dt}] \, dt + i \int_{t_0}^{t_1} [v \frac{dx}{dt} + u \frac{dy}{dt}] \, dt
\]

These are real integrals:

\[ \int_{t_0}^{t_1} u \, dt - \int_{t_0}^{t_1} v \, dt + i \int_{t_0}^{t_1} [u(\frac{d}{dt}) \, dt + i \int_{t_0}^{t_1} [v(\frac{d}{dt}) \, dt] \]
An inequality involving contour integrals.

\[ \oint_C f(z) \, dz = \lim_{\max|\Delta z_i| \to 0} \sum_{i} f(z_i) \Delta z_i \]

\[ |z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n| \]

\[ \sum |f(z_i)| \Delta z_i \leq \sum |f(z_i)| \, |\Delta z_i| \]

\[ \left| \oint_C f(z) \, dz \right| \leq \oint_C |f(z)| \, |dz| \]

Suppose that on \( C \),

\[ |f(z)| \leq M \]

\[ dz = (x + iy) \, dt \]

\[ |dz| = \sqrt{x^2 + y^2} \, dt \]

\[ \left| \oint_C f(z) \, dz \right| \leq M \oint_C \sqrt{x^2 + y^2} \, dt \]

where \( l \) is the length of \( C \).
Simply connected domain

If \( f(z) \) is an analytic function that is regular in a simply connected domain \( D \), and at the boundary of \( D \) formed by curve \( C \), then,

\[
\oint_C f(z) \, dz = 0
\]

Goursat (1900) generalized this theorem for \( f(z) \) continuous on \( C \).
Pollard (1923) proved it also for \( f(z) \) existing on \( C \).

Connection with Stokes' theorem.

\[
\int_{\mathbb{R}} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, ds = \oint_C \mathbf{F} \cdot d\mathbf{r}
\]

If \( \mathbf{F} \) is a two-dimensional vector field,

\[
\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} = u \mathbf{i} + v \mathbf{j}
\]

If \( f(z) = u + iv \) \( \Rightarrow \int_{\mathbb{R}} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, ds = \text{Im} \left[ \oint_C f(z) \, dz \right] \)

If analytic \( \Rightarrow \nabla \times \mathbf{F} = 0 \) (from C-R conditions) and \( \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \)
Multiply-connected domain

\[\oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \oint_{C_4} f(z)\,dz = 0\]

\[\Rightarrow \oint_{C_2} f(z)\,dz = -\oint_{C_1} f(z)\,dz\]

if \( f(z) \) is analytic and regular inside the domain between \( C_1 \) and \( C_2 \).

Lecture 7, 9/25

Wed. Oct. 16, no class.
Friday additional class Nov. 1

Cauchy integral (Cauchy 2nd theorem).

If \( f(z) \) is an analytic function, regular on a simply connected region \( D \), and the boundary, \( C_1 \) of \( D \), and if \( a \) is any interior point of \( D \), then,
\[ f(\alpha) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-\alpha} \, dz \]

where the integral must be evaluated along \( C \) in the counterclockwise direction.

It is evident that \( \oint_C \frac{f(z)}{z-\alpha} \, dz = 0 \) if \( \alpha \) is outside \( D \).

\[
\frac{1}{2\pi i} \left( \oint_{C_1} \frac{f(z)}{z-\alpha} \, dz + \oint_{C_2} \frac{f(z)}{z-\alpha} \, dz \right) = f(\alpha) \quad \text{if} \quad \alpha \quad \text{is inside the shaded region}
\]

**Cauchy's Formulas for the Derivatives of a Regular Analytic Function**.

\[
f'(\alpha) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-\alpha)^2} \, dz
\]

**Theorem**. If \( f(z) \) is analytic and regular in \( D \) and on the boundary of \( D \), formed by a closed curve \( C \), then \( f(z) \) has derivatives at a point \( \alpha \) inside \( D \):

\[
f^{(n)}(\alpha) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-\alpha)^{n+1}} \, dz, \quad n \in \mathbb{N}
\]
Cauchy's Inequalities.

\[ |f^{(n)}(a)| = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} \, dz \]

\( f(z) \) is continuous on \( C \) (since it is regular) and then:

\[ |f(z)| \leq M(R) \text{ on } C \]

\[ |f^{(n)}(z)| \leq \frac{n!}{2\pi} \left( \frac{M(R)}{R^{n+1}} \right) 2\pi R \]

\[ |f^{(n)}(z)| \leq \frac{n! M(R)}{R^n} \quad n=0, 1, 2, \ldots \]

Liouville's Theorem

If \( f(z) \) is integral and bounded, namely,

\[ \exists M \cdot |f(z)| \leq M \text{ for all } z \]

\[ |f(z)| \leq \frac{M}{R} \forall R \Rightarrow f(z) = 0 , \forall z \]

then \( f(z) \) is constant.

Therefore \( f(z) \) is unbounded, and for every \( \exists > 0 \), every \( z \) there is a \( z' \), such that

\[ |z - z'| < \exists \text{ and } |f(z')| \geq f(z) \]
**Morera's theorem.**

If \( \oint_C f(z) \, dz = 0 \) for every closed curve on \( C \), then \( f(z) \) is analytic on \( D \).

**Taylor's Theorem.**

Let \( f(z) \) be an analytic function that is regular in \( |z-a|<r \). Then at any point \( z \in \{ |z-a|<r \} \), \( f(z) \) can be represented as a convergent series of powers of \( (z-a) \):

\[
f(z) = \sum_{n=0}^{\infty} A_n (z-a)^n
\]

where:
\[
A_n = \frac{1}{2 \pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} \, dz = \frac{1}{n!} f^{(n)}(a)
\]

\( C = \{ z / |z-a|=r \} \)

If \( M(r) \) is the max. of \( |f(z)| \) on \( C \),

\[
|A_n| \leq \frac{M(r)}{r^n}
\]

**Theorem.**

A function which is regular everywhere, including the point at infinity, is a constant.
Proof:
\[ f(z) = \sum_{n=0}^{\infty} A_n z^n \quad \text{[} R = \infty \text{]} \]
\[ \Rightarrow f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} A_n \left(\frac{1}{z}\right)^n \]

when \( z \to 0 \), we get \( A_n = 0 \) \( \forall n \neq 0 \)
\[ \Rightarrow f(z) = A_0 \]

**THE ZEROS OF A REGULAR ANALYTIC FUNCTION.**

Let \( f(z) \) be analytic and regular in \( |z-a| \leq r \)

\[ f(z) = \sum_{n=0}^{\infty} A_n (z-a)^n \]

If \( f(a) = 0 \) \( \Rightarrow A_0 = 0 \), and \( a \) is called a zero of \( f(z) \).

If \( A_0 = A_1 = \cdots = A_{N-1} = 0 \)
\[ \Rightarrow f(z) = \sum_{n=N}^{\infty} A_n (z-a)^n = (z-a)^N \sum_{k=0}^{\infty} A_{N+k} (z-a)^k \]

\[ f(z) = (z-a)^N \phi(z) \quad \text{where} \quad \phi(a) \neq 0 \]

\( a \) is called a zero of order \( N \).
It can be shown that $\phi(z)$ is analytic and regular in the domain $|z-c|<r$.

$$A_k = \frac{1}{k!} f^{(k)}(c)$$

Therefore, if $c$ is a zero of order $N$, 

$$f(c)=f'(c)=\ldots=f^{(N-1)}(c)=0, \quad f^{(N)}(c) \neq 0$$

**THEOREM.**

Let $f(z)$ be an analytic function which is regular in a neighborhood of a point $z=c$ and $f(c)=0$; then, unless $f(z)=0$, there exists a neighborhood of $z=c$ where $f(z)$ have no other zeroes than $z=c$. That is, the zeroes of an analytic function are isolated.

**Proof:** If $f(z) \neq 0$

$$f(z) = (z-c)^N \phi(z)$$

$\phi(z)$ is continuous at $z=c$.

$\exists \varepsilon > 0, \exists \delta > |\phi(c)-\phi(z)| > 0, \forall |z-c| < \delta$

Choose $\varepsilon = \frac{1}{2} |\phi(c)| \neq 0$

$$|\phi(z)| \geq |\phi(c)| - |\phi(c)-\phi(z)| > \varepsilon$$

$$\Rightarrow |f(z)| = |z-c|^N |\phi(z)| > 0 \text{ for } z \neq c \text{ and } |z-c| < \delta.$$
THEOREM.

If \( f(z) \) is an analytic function regular in a domain, and if \( z_1, z_2, \ldots, z_n \) are zeroes of \( f(z) \) having as a limiting point in the interior of \( D \), then \( f(z) = 0 \) in all the domain.

Laurent's theorem (1843)

If \( f(z) \) is analytic and regular in the closed annulus \( r_2 \leq |z - a| \leq r_1 \), it may be represented in the form:

\[
f(z) = \sum_{n=0}^{\infty} A_n (z-a)^n + \sum_{n=1}^{\infty} \frac{B_n}{(z-a)^n}
\]

\[
A_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{(z-a)^{n+1}} \, dz
\]

\[
B_n = \frac{1}{2\pi i} \oint_{C_2} f(z) (z-a)^{n-1} \, dz
\]

\( N \geq 0 \), \( r_2 \leq |z-a| < r_1 \)
\[ f_1(z) = \sum_{n=0}^{\infty} A_n (z-a)^n \]
\[ f_2(z) = \sum_{n=1}^{\infty} B_n \frac{1}{(z-a)^n} \]

Obviously, \( f(z) \) converges at \( r_2 < |z-a| < r_1 \), if \( z=a \) is the only singularity of \( f(z) \) inside \( |z-a| < r_1 \), then the expansion is valid for every \( z \) such that \( |z-a| < r_1 \) and \( z \neq a \).

If \( f(z) \) has no singularities at all, \( f(z)(z-a)^{n-1} \) is analytic, and,
\[ B_n = \frac{1}{2\pi i} \oint_{C_2} f(z)(z-a)^{n-1} \, dz = 0 \]
and \( A_n = \frac{1}{n!} f^{(n)}(a) \)

**Classification of Isolated Singularities**

If \( a \) is the only singularity in the domain \( |z-a| \leq r \),
\[ f(z) = f_1(z) + f_2(z) \quad \text{at} \quad z \cdot 0 < |z-a| < r \]
\[ \uparrow \quad \uparrow \]
Regular \quad Principal
part \quad part
1. $B_n = 0$  \( n = 1, 2, \ldots \)

   \[ \lim_{z \to a} f(z) = \left( \sum_{n=0}^{\infty} \frac{A_n}{(z-a)^n} \right)_{z=a} = A_0 \]

   $f(a)$ does not exist, but can be defined as $A_0$, thus giving a new function which is regular in the neighborhood.

2. The principal part contains a finite number of terms.
   Then, \( \exists N \cdot B_n \neq 0, B_m = 0 \ \forall m > N \).

   $a = \text{POLE OF ORDER } N$.

3. The principal part is an infinite series.
   \( \text{ESSENTIAL SINGULARITY} \).

   **POINT AT INFINITY.**

   If $f(z)$ is regular in the neighborhood of the point at infinity but not necessarily at the point at infinity.

   \[ f(\frac{1}{z}) = \sum_{n=0}^{\infty} \frac{A_n}{z^n} + \sum_{n=1}^{\infty} B_n z^n \]

   1. $A_m = 0$  \( \forall m \neq 0 \).
      Removable singularity at infinity

   2. $A_n = 0$, \( \forall n > N \). POLE OF ORDER $N$ at infinity.
3. The A-series has an infinite number of terms. ESSENTIAL SINGULARITY at infinity.

Lecture 9, 10/2.

THEOREMS about singular points.

$z = a$ is a pole of order $N$ of $f(z)$

$$f(z) = \frac{B_N}{(z-a)^N} + \frac{B_{N-1}}{(z-a)^{N-1}} + \ldots + \frac{B_1}{(z-a)} + A_0 + A_1(z-a)^1 + \ldots$$

$$f(z) = \frac{1}{(z-a)^N} \phi(z)$$

where $\phi(z)$ is regular in $|z-a| < R$, and $\phi(a) = B_N \neq 0$.

$$\Rightarrow \frac{1}{f(z)} = \frac{(z-a)^N}{\phi(z)}$$

is analytic in $|z-a| < R$.

1. and since $z = a$ is not a zero of $\phi(z)$

$1/f(z)$ is also regular in some neighborhood of $z = a$; furthermore, it has a zero of order $N$ in $z = a$.

Since $\phi(z)$ is continuous in $|z-a| < R$,

$$\exists \varepsilon > 0 : |\phi(z)| > \frac{1}{2} |\phi(a)| = \frac{1}{2} |B_N|, \forall 1z-a| < \varepsilon$$

$$\Rightarrow |f(z)| = \frac{1}{|\phi(z)||z-a|^N} > \frac{1|B_N|}{2|z-a|^N}$$

2. If $f(z)$ has a pole at $z = a$, it is not bound in any neighborhood of $a$. 
\[ K > 0; \exists \delta \cdot |f(z)| > K, \forall |z| < \delta \]

Then we say \(|f(z)| \to \infty\) as \(z \to \infty\)

\(3\) **WEIERSTRASS' THEOREM.**

In every neighborhood of an isolated essential singularity there exists a point (an infinite number) at which \(f(z)\) differs at least as we went from an arbitrary given value.

\(\exists \delta > 0, \epsilon > 0, C; \exists z_0 \cdot |f(z_0) - C| > \epsilon, \forall |z| < \delta\)

For the proof, see Copson pp 81-82.

\(4\) **PICCARD'S THEOREM (2nd)**

In every neighborhood of an isolated essential singularity there exists a point (actually an infinity of such points) at which \(f(z)\) attains any arbitrarily chosen value with at most one exception.

Proof: E.C. Titchmarsh, Theory of funct. pp 283-284

\(5\) **PICCARD'S THEOREM (1st)**

An integral function which is not a constant attains any value, with one possible exception, at least once.

Copson pp 438-440
LIMITING POINTS OF ZEROES.

If $z_0$ is the limiting point of the zeroes of a function $f(z)$ which is analytic and regular in $0 < |z - z_0| < r$, then $z = z_0$ is an isolated essential singularity of $f(z)$, unless $f(z)$ vanishes identically in $0 < |z - z_0| < r$.

Example:

$$f(z) = \sin \frac{1}{z}$$

Zeroes: \quad \frac{1}{z} = n\pi \quad z = \frac{1}{n\pi} \quad n = \pm 1, \pm 2, \pm 3, \ldots$

$$\frac{1}{n\pi} \to 0 \quad \text{as} \quad n \to \infty.$$ $\sin \frac{1}{z} = \frac{1}{z} - \frac{1}{3!} \left( \frac{1}{z^2} \right) + \frac{1}{5!} \left( \frac{1}{z^3} \right) - \ldots$

$\Rightarrow$ $z = 0$ is an isolated essential singularity.

The singularity at infinity

Every integral function has a singularity at infinity. Polynomials have a pole of degree $N$ at the point at infinity, and no other singularity.
THEOREMS

1. If $f(z)$ is analytic and regular in $D$, its real and imaginary parts have no extrema in the interior of $D$, unless $f(z)$ is a constant.

2. If $f(z)$ is analytic and regular in $D$, $\text{mod}(f(z))$ has no maximum in the interior of $D$, unless $f(z)$ is a constant. If $f(z)$ is a constant, the only possible minimum value of $f(z)$ is zero.


UNIFORM CONVERGENCE

$$a_n = A_n z^n$$

$$S(z) = a_0(z) + a_1(z) + a_2(z) + \cdots$$

Even if $a_n(z)$ is continuous for every $n$, $S(z)$ may not be continuous.

**Example:**

$$S(x) = x^2 + \frac{x^2}{(1+x^2)} + \frac{x^2}{(1+x^2)^2} + \cdots$$

$$a_n(x) = \frac{x^2}{(1+x^2)^n}$$ is continuous for $x^n$ and $x \in \mathbb{R}$. 
for \( x=0 \), \( S(x=0)=0 \)

\[ x \neq 0, \quad S(x) = x^2 \left( \frac{1}{1 - \frac{1}{1+x^2}} \right) = (x^2+1) \]

\[ \Rightarrow \lim_{x \to 0} S(x) = 1 \neq S(0) \]

Lecture 10, 10/7/85

\[ \{S_n(x)\} \]

\[ S_n(x) = \sum_{k=0}^{n} a_n(x) \]

Assume that the \( a_n(x) \) are continuous functions; i.e., for any particular value of \( x \) (\( a \leq x \leq b \)), \( |a_n(x) - a_n(x')| < \varepsilon \), \( \forall |x-x'| < \delta \).

We also assume that \( S_n(x) \) converges. However, given \( \varepsilon \), \( n_0 \) for which \( |S-S_n| < \varepsilon \), \( n \geq n_0 \) may depend on \( x \), and it may be impossible to find a value \( n_0 \) for which \( |S-S_n| < \varepsilon \) for every \( x \) and every \( n > n_0 \).

**Definition.**

A sequence \( \{S_n(z)\} \) is said to converge uniformly to the value \( S(z) \) in the domain \( D \), if

\[ \forall \varepsilon > 0, \exists n_0 \text{ (independent of } z) \quad |S(z)-S_n(z)| < \varepsilon \quad \forall \quad n \geq n_0(\varepsilon) \]
We define the remainders $R_n(x)$ as,

$$R_n(x) = S(x) - S_n(x)$$

**PRINCIPLE OF UNIFORM CONVERGENCE**

A sequence of functions $\{S_n(x)\}$ is uniformly convergent in a bounded closed domain $D$, if and only if,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \ni |S_{n+p} - S_n| < \varepsilon, \forall p$$

**Properties of uniformly convergent series**

1. **THEOREM**

   If each term $a_n(x)$ of an infinite series
   $$\sum_{n=0}^{\infty} a_n(x)$$
   is continuous in a bounded closed domain $D$, and if the series converges uniformly in $D$, then $S(x)$ is also continuous on $D$.

2. If $\{a_n(x)\}$ are continuous and $S(x)$ is uniformly convergent,
   $$\int S(x) \, dz = \sum_{n=0}^{\infty} \int a_n(x) \, dz$$

3. If $a_n(x)$ is analytic and regular in $\mathbb{C}$, and $S(x)$ converges uniformly in every closed domain of $\mathbb{C}$, then $S(x)$ is also analytic and regular in $\mathbb{C}$. 
Moreover, \( S(z) = \sum_{n=0}^{\infty} a_n(z) \) is also uniformly convergent in every closed domain \( D \) within \( C \).

**WEIERSTRASS M-TEST FOR UNIFORM CONVERGENCE.**

A series \( \sum_{n=0}^{\infty} a_n(z) \) converges uniformly in a bounded, closed domain \( D \), if \( |a_n(z)| \leq M_n \) for all \( z \in D \), where \( M_n \) are independent of \( z \), and \( \sum_{n=0}^{\infty} M_n \) is convergent.

The Taylor and Laurent series converge uniformly in any closed domain within their domains of convergence.

**MANY-VALUED FUNCTIONS. RIEMANN SURFACES**

Example: \( w = \sqrt{z} \)

\[ z = re^{i\theta} \]

\[ W_1 = +\sqrt{r}e^{i\theta/2} \quad W_2 = +\sqrt{r}e^{i(\theta/2 + \pi)} = -W_1 \]
\( W = z^{1/2} \) has no continuous one-valued solutions in the \( z \)-plane.

After two cycles, \( \theta = 4\pi \), we get again the same value \( +\sqrt{r} \) for \( W \), continuous. Then we could have still a one-valued function if we extend the concept of complex plane. \( W_1, W_2 \) are called BRANCHES.

\( z = 0 \) is called a branch point. \( W \) is one valued at \( z = 0 \), but as we go around that point, we get two branches. The point at infinity, is also a branch point. The line \( (0,\infty) \) is the branch line. The choice of the branch line is arbitrary.

Lecture 11th, 10/9/85

Algebraic branch point

Cut Argand plane

Riemann Surface \((\mathcal{R}_1 \cup \mathcal{R}_2)\)
At every point of the Riemann surface, except at the branch point \( z = 0 \), \( w \) constitutes a regular, single valued, analytic function.

**LOGARITHMIC BRANCH POINT**

\( \log z, \quad z^{1+i}, \quad \arccos z \)

In these cases, the Riemann surface has an infinite number of branches.

**Examples.**

1. \( W = \sqrt{z} \) where \( n \) is a positive integer.

\( z = 0 \) is a branch point

\( z = re^{i\theta} \)

By de Moivre's theorem,

\[ W_1 = \sqrt{r} e^{i\theta/n} \quad 0 \leq \theta \leq 2\pi \]

\[ W_n = \sqrt{r} e^{i \left( \frac{\theta + (n-1)2\pi}{n} \right)} \]

\[
\begin{array}{c}
T_{1n} \\
T_{1(n-1)} \\
T_{13} \\
T_{12} \\
T_{11}
\end{array}
\]

\[
\begin{array}{c}
W_n \\
W_{n-1} \\
W_3 \\
W_2 \\
W_1
\end{array}
\]
\( W = \log z = \log r + i \theta \quad (n \in \mathbb{Z}) \)

Branch points: \( z = 0 \)

Point at infinity:

\[
\log \frac{1}{z} = \log \frac{1}{re^{i\theta}} = \log (\frac{1}{r^m}) - i\theta' \\
= -\log r - i\theta'
\]

\(-\pi < \theta_0 \leq \pi\) principal value of \( \theta \).

Branches:

\[
W_m = \log z = \log r + i(\theta_0 + 2m\pi) \quad m = 0, \pm 1, \pm 2, \ldots.
\]

There is an infinite number of branches.

Since we chose \( \theta_0 \) between \(-\pi\) and \( \pi\), the branch cut will be the negative real semiaxis.

\[
\log z = \log |z| + i\theta_0
\]

(principal value of \( \log \))

\[
\log |z| = \int_1^z \frac{dz}{z}, \quad \text{if} \ z \ \text{is real}
\]

\[
\log z = \int_1^z \frac{dz}{z}, \quad \text{if} \ z \ \text{is complex, the integral must be calculated taking as many turns as it is necessary to get to the value} \ z \ \text{(depending on which branch \( z \) is)}
\]
3. \[ W = \sqrt{(z-a)(z-b)} \]

Branch points: \( z = a, \ z = b \)

\[
W = \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)}
\]

\[
W_1 = +\sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2}
\]

\[
W_2 = -\sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2}
\]

Suppose we go along a complete circuit around \( z = a \), without going around \( b \):

\[ W_1 \rightarrow \sqrt{r_1 r_2} e^{i(\theta_1 + 2\pi + \theta_2)/2} = W_2 \]

If we now go around \( a \) and \( b \),

\[ W_1 \rightarrow \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2 + 4\pi)/2} = W_1 \]

Point at infinity.

\[ z = \frac{1}{\bar{z}}, \quad z' = r e^{i\theta} \]

\[ W = \sqrt{\left( \frac{e^{-i\theta}}{r} - a \right) \left( \frac{e^{-i\theta}}{r} - b \right)} \approx \sqrt{\frac{e^{-2i\theta}}{r^2}} \] if \( r \approx 0 \)

\[ W_1 \approx \frac{1}{r} e^{-i\theta}, \quad W_2 \approx \frac{1}{r} e^{-i\theta} \]
after a turn around $z'=0$,
\[ W_1 \rightarrow W_1 e^{2\pi i} = W_1, \]
\[ W_2 \rightarrow W_2 e^{2\pi i} = W_2 \]
then, the point at infinity is not a branch point.

\[ W = \sqrt{(z-a_1)(z-a_2) \cdots (z-a_n)} \]

Branch points:
\[ a_1, a_2, \ldots, a_n \]
Point at infinity:
\[ W\left(\frac{1}{z'}\right) = \sqrt{\left(\frac{e^{-i\theta_1}}{r_1} - a_1\right) \cdots \left(\frac{e^{-i\theta_n}}{r_n} - a_n\right)} \]
\[ W\left(\frac{1}{z'}\right) \approx \sqrt{\frac{1}{r_{\infty}} e^{-i\theta_{\infty}}} \quad r_{\infty} \approx 0 \]
\[ W_1 \approx \left(\frac{1}{r_1}\right)^{\frac{n}{2}} e^{-i\theta/2} \quad W_2 \approx \left(\frac{1}{r_n}\right)^{\frac{n}{2}} e^{-i\theta/2} \]
\[ e^{i\pi(n/2)} = \begin{cases} +1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases} \]
Thus, if $n$ is even the point at infinity
is not a branch point, and if \( n \) is odd, the point at infinity is a branch point.

\[ y \\
\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\end{array} \\
\begin{array}{c}
\alpha_6 \\
\end{array} \\
x 
\]

\[ W = \sqrt[3]{z-a} \sqrt[3]{(z-b)} \]

Branch points have many applications in physical problems like scattering and diffraction.

References:
- Morse & Feshbach, V.I
- K. Knopp, Theory of functions
- P. Dines, Taylor series.
ANALYTIC CONTINUATION

\[ f_1(z) = 1 + z + z^2 + \ldots \quad \text{Converges for } |z| < 1 \]

\[ f_2(z) = \int_0^\infty e^{-t(1-z)} \, dt = \frac{1}{1-z} \quad \text{with } \Re(z) < 1 \]

(if \( \Re(z) > 1 \), it diverges)

\[ f_3(z) = \frac{1}{1-z} \quad \forall z \neq 1 \]

\( f_1(z), f_2(z) \) analytic and regular in \( D_1 \) and \( D_2 \) respectively. \( D_2 = D_1 \cap D_2 \neq \emptyset \). Suppose \( f_1 = f_2 \) for every point in \( D_2 \); we can define,

\[ F(z) = \begin{cases} f_1(z), & z \in D_1, \\ f_2(z), & z \in D_2 \end{cases} \]

\( F(z) \) is analytic and regular in \( D = D_1 \cup D_2 \).

Is there any other function \( F(z) \) analytic and equal to \( f_i \) in \( D_i \)?

THEOREM:

If \( f_i \) is a given function analytic and regular in \( D_i \), and if also a domain \( D_2 \) is given, which has a region \( D_2 \) in common with \( D_1 \), there is at most one function which is analytic and regular in \( D_1 + D_2 \) and is equal to \( f_i(z) \) in \( D_i \).
This is called **PRINCIPLE OF ANALYTIC CONTINUATION**.

**Proof**: \( F(z), G(z) \) analytic and regular in \( D \).

Assume \( F(z) = G(z) \) in a subdomain of \( D \).

\[
\Phi(z) = F(z) - G(z)
\]

is analytic and regular in \( D \), and

\[
\Phi(z) = 0 \text{ in the subdomain}
\]

**THEOREM**.

If \( \Phi(z) \) is analytic and regular in \( D \), and \( z_1, z_2, \ldots \) is a sequence of zeros of \( \Phi(z) \) in \( D \), if the sequence has a limit in \( D \), then \( \Phi(z) = 0 \) in \( D \).

**Proof**:

\[
\Phi = \sum A_n (z - z_i)^n
\]

Along the line of zeros, it is trivial.

\[
A_n = \frac{1}{n!} \Phi^{(n)}(z_i) = 0, \quad n
\]

We have assumed here that \( \Phi(z) = 0 \) for a segment of line; i.e., we have assumed \( F(z) = G(z) \) for a segment of line.

This means that there is only one function in an analytic and with a certain value throughout a subdomain of \( D \).
The fact that there is just one analytic function in $D$ which has a given value at a subregion, does not imply that if we approach a point in $D$, we get the same value for $f(z)$ independently of the path we take. (There may be different branches).

\[ D_{123} = D_1 \cap D_2 \cap D_3 = \emptyset \]

\[ \circlearrowright f_2 = f_1 \text{ in } D_1 \cap D_2 \quad \Rightarrow \quad f_2 = f_3 \text{ in } D_{123} \]
\[ f_3 = f_1 \text{ in } D_1 \cap D_3 \]
\[ \circlearrowright f_2 = f_3 \text{ in } D_{23} \]

Therefore, the analytic continuation may be a multi-valued function. For example, if in the process of continuation we go around a branch point, we get a two-valued function.
\[ f_1(z) = 1 + z + z^2 + \ldots, \quad |z| < 1 \]
\[ f_2(z) = \int_0^\infty e^{-t(1-z)} \, dt \quad \text{Re}(z) < 1 \]

\[ F(z) = \frac{1}{1-z}, \quad z \neq 1 \] is the analytic continuation of \( f_1 \) and \( f_2 \).

Three ways of defining general analytic functions:

(i) \( f(z) = \sum_{n=0}^\infty A_n (z-a)^n \)

(ii) \( f(a), f'(a), f''(a), \ldots \)

(iii) give the value of \( f(z) \) along a segment of line (or a sequence of points and its limit)

\[ \text{TEST.} \]

2. \( \sum_{n=0}^\infty \frac{(n!)^2}{(2n)!} z^n \quad R = 4 \)

\[ |z| = 4, \quad |\Theta_n| \to 1 \]

\[ \Rightarrow \text{the series diverges.} \]

5. \( S(x) = \sum_{n=1}^\infty \frac{x^n}{n2^n} \) is it uniformly convergent?

5. Find the singularities of:

\[ f(z) = \frac{1 - \cos z}{z^2(z^2 + 9)} \quad \text{ans.} \ 0, \pm 3i \]

\[ f(z) = \sec \frac{\pi}{z} = \frac{1}{\cos \frac{\pi}{z}} \quad 0, \frac{1}{n + \frac{1}{2}}, n = 0, \pm 1, \pm 2, \ldots \]

\( \text{Non isolated, essential} \)
$P_n(x) = \text{polynomial of degree } n.$

If $P_n$ would not have any zero, $\frac{1}{P_n}$ is continuous inside any circle of arbitrary radius $R$ and therefore it is bounded inside the circle.

Outside the circle (neighborhood of the point at infinity) $\frac{1}{P_n}$ is also bounded, since

$$\frac{1}{P_n} \to 0 \quad \text{as } |z| \to \infty$$

Lecture 13, Oct. 28

**Natural Boundary.**

Example: $f(z) = 1 + z^2 + z^4 + z^8 + \ldots$

converges for $|z| < 1$

$f(z^2) = 1 + z^4 + z^8 + z^{16} + \ldots$

$= f(z) - z^2 \quad \Rightarrow f(z) = f(z^2) + z^2.$

Since $f(z)$ diverges at $z = 1$,

$f(z) \to \infty \quad \text{as } z^2 \to 1, \text{i.e.: } z = \pm 1$

in the same way, $f(z^4) = f(z^2) - z^4$ diverges at $z = \pm 1$

$\Rightarrow f(z)$ diverges at every point on the circle $|z| = 1.$
Question: What about an irrational value for $z$? At that $z$ we can not prove that $f(z)$ diverges. But between two irrational numbers there is always a rational one, and thus it is not possible to find an interval where $f(z)$ is analytic, and we can not cross the boundary.

ANALYTIC CONTINUATION BY MEANS OF POWER SERIES.

$$f(z) = \sum_{n=0}^{\infty} A_n (z-z_0)^n$$

If $R_1 \geq R_0 - |z_i - z_0|$, then the circle $C_i$ includes some points not in $C_0$.

This method of continuation is due to Weierstrass & Mergely.

REGULAR POINT.

A regular point of a one-valued analytic function is any point interior to one of the circles which have been or could be used to carry out the continuation from the original element.
SINGULAR POINT
A point which is not regular, but is a limiting point of regular points is said to be a singular point of the complete analytic function.

THEOREM.
On the circle of convergence there is at least one singular point of the analytic function.

Proof: Titchmarsh. Theory of functions

Lecture 14, 10/30.

MANY-VALUED FUNCTIONS.
\[ f(z) = \frac{1}{\log z} \quad z = re^{i\theta} \]
\[ f(z) = \frac{1}{\log r + i(\theta_0 + 2m\pi)} \quad m = 0, \pm 1, \pm 2, \ldots \]

\[ z = 1, \quad r = 1, \quad \theta_0 = 0. \]
\[ f(1) = \frac{1}{2im\pi} \]

Singularity at \( z = 1 \) when \( m = 0 \).

\( z = 1 \) is a singular point of the branch \( m = 0 \), but is a regular point for all the other branches.

Copson, p. 85
1. Schwarz reflection principle.
(schwarz principle of symmetry).

\[ f(z) \text{ analytic and regular in } D. \]
Continuous on \( T \) and on the segment \( AB \).
\[ f(z) \text{ real on the segment } AB. \]
\[ \Rightarrow f(z^*) = f(z) \quad \text{and } f(z) \text{ it's been continued in the reflected domain } D^*. \]

2. USE OF FUNCTIONAL EQUATIONS.

**Gamma function:**
\[ T'(z) = \int_0^\infty e^{-t} t^{z-1} dt \]

\( T'(z) \) is a multivalued function.

**Principle values:**
\[ t^{z-1} \rightarrow e^{(z-1) \log t} \]

It can be proved that \( T'(z) \) converges for \( Re(z) > 0 \) (Copson pp.206-207)

\[ T'(z+1) = T'(z) \quad z \]

\[ D_1 : Re(z) > 0 \]
\[ D_2 : -1 < Re(z) < \frac{1}{2} \]

\[ f_2(z) = T'(z+1) = \frac{f_1(z+1)}{z} = \frac{f_1(z)}{z} \]
in \( D_2 \)
The only problem is at \( z = 0 \).
In the same way, the function can be continued into the domain \(-n < Rz < -(n-1)+\frac{1}{2}\).

\( z = 0 \) is a branch point.
\( z = -1, -2, -3, \ldots \) are simple poles.

\[
T'(1) = \int_0^\infty e^{-t} \, dt = 1 \\
\Rightarrow T'(n) = (n-1)!
\]

**Eulerian Integral of the First Kind.**

Beta function: \( B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} \, dt \)

we will always consider the principle values of \( t^{p-1} \) \((t)^{q-1}\) only.

Converges for \( Rp > 0, Rq > 0 \)

(i) \( B(p,q) = B(q,p) \)

(ii) \( B(p,q) = \frac{T(p) \, T(q)}{T(p+q)} \)

(iii) \( B(p,q) = 2 \int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta \cos^{2q-1}\theta \, d\theta \)

3. **Borel's Method of Continuation.**

\[
f(z) = \sum_{n=0}^{\infty} A_n z^n
\]

\( R \): radius of convergence
Let, \[ \phi(z) = \sum_{n=0}^{\infty} \frac{A_n}{n!} z^n \]

\[ F(z) = \int_{0}^{\infty} e^{-t} \phi(zt) \, dt \]

\( F(z) \) is an analytic continuation of \( f(z) \) across any region where there are no singularities.

\( F(z) \): one-sided Laplace transform.

Proof: Titchmarsh, p.164

Whittaker, Watson p.141

Example.
\[ f(z) = 1 + z + z^2 + z^3 + \ldots \]
\[ R = 1 \]
\[ \phi(z) = 1 + \frac{z}{1!} + \frac{1}{2!} z^2 + \ldots = e^z \]

\[ F(z) = \int_{0}^{\infty} e^{-t} \phi(zt) \, dt \]

\[ = \int_{0}^{\infty} e^{-t} e^{zt} \, dt = \frac{1}{z-1} e^{t(z-1)} \bigg|_{0}^{\infty} \]

If \( Rz \leq 1 \), \( e^{t(z-1)} \to 0 \) as \( t \to \infty \)

\[ F(z) = \frac{1}{1-z} \), \( Rz < 1 \) (if \( x < 1 \))
CAUCHY'S RESIDUE THEOREM.

Residue.
Suppose \( z = a \) is an isolated singularity of the function \( f(z) \).

\[ f(z) = \phi(z) + \frac{B_1}{z-a} + \frac{B_2}{(z-a)^2} + \cdots \]

where \( \phi(z) \) is analytic at \( z = a \).

\( B_1 \) is called the residue of \( f(z) \) in \( a \).

\[ B_1 = \frac{1}{2\pi i} \oint_C f(z) \, dz \]

THEOREM.

Let \( f(z) \) a function analytic and single-valued in a closed domain \( D \). If \( C \) is any closed simple curve in \( D \), then.

\[ \oint_C f(z) \, dz = 2\pi i \sum_{j=1}^{n} r_j \]

where \( r_j \) is the residue of the function \( f(z) \) on its \( j^{th} \) singularity, and \( f(z) \) has \( n \) singularities inside \( C \). The integral is evaluated in the anticlockwise sense.
Proof:

Let us draw a circle $C_j$ around each of the $j$'th singularities, in such a way that inside each circle there is only one singular point $(a_j)$.

Since $f(z)$ is analytic in the region between $C$ and $x_1, x_2, \ldots, x_n$,

$$\oint_{C} f(z) \, dz = \sum_{j=1}^{n} \oint_{C_j} f(z) \, dz = \sum_{j=1}^{n} (2\pi i x_j)$$

If $z = a$ is a pole of order $n$,

$$g(z) = (z-a)^n f(z)$$

is analytic in $z = a$.

$$f(z) = \phi(z) + \frac{B_1}{(z-a)} + \ldots + \frac{B_n}{(z-a)^n}$$

$$g(z) = (z-a)^n \phi(z) + B_1 (z-a)^{n-1} + \ldots + B_n$$

\[\text{powers } \geq n \quad \text{analytic part.}\]

\[\Rightarrow B_1 = \frac{1}{(n-1)!} g^{(n-1)}(a)\]

$$R(a) = B_1 = \frac{1}{(n-1)!} \lim_{z \to a} \left\{ \frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)] \right\}$$
THE NUMBER OF ZEROS OF ANALYTIC FUNCTIONS.

THEOREM.

Let \( f(z) \) be an analytic function in the domain \( D \), and \( C \) a closed curve inside \( D \). If \( f(z) \) has only pole singularities (or not at all), and \( f(z) \) to along the curve \( C \) and \( f(z) \) is regular along \( C \),

\[
\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} \, dz = N_z - N_p
\]

where \( N_z \) is the number of zeros and \( N_p \) is the number of poles.

A pole of order \( n \) is counted as \( n \) poles, and so a zero of order \( n \).

Proof: \( \frac{f'(z)}{f(z)} \) has singularities only at the points \( z \) where either \( f(z) = 0 \) or \( f(z) \) has a pole.

1) \( z \) is a zero of order \( m \) of \( f(z) \)

\[ \Rightarrow f(z) = (z-\alpha)^m \, g(z) \]

where \( g(z) \) is regular in some neighborhood of \( z=\alpha \), and \( g(\alpha) \neq 0 \) in some neighborhood of \( z=\alpha \).

\[
\log f(z) = m \log (z-\alpha) + \log g(z)
\]

well behaved

\[
\frac{f'(z)}{f(z)} = \frac{m}{z-\alpha} + \frac{g'(z)}{g(z)}
\]
and so, \( \frac{f'(z)}{f(z)} \) has only one singularity in the neighborhood, which is a pole of degree 1 (SIMPLE POLE), with residue \( m \).

Therefore the sum of residues of \( \frac{f'(z)}{f(z)} \) at the zeros of \( f(z) \) is \( N_z \).

(i) \( z \) is a pole of order \( m \).

\[ f(z) = (z-a)^m g(z) \]

where \( g(z) \) is regular and different from zero in a neighborhood of \( z=a \).

\[ \log f(z) = -m \log (z-a) + \log g(z) \]

\[ \frac{f'(z)}{f(z)} = \frac{-m}{z-a} + \frac{g'(z)}{g(z)} \]

\( \Rightarrow \) The sum of residues of \( \frac{f'(z)}{f(z)} \) at the poles is \( -N_p \).

And the theorem follows.

The theorem can also be written as,

\[ \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi i} [\log f(z)]_C \]

\[ = \frac{1}{2\pi i} [\log |f(z)| + i \arg f(z)]_C \]

\[ \Rightarrow N_z - N_p = \frac{1}{2\pi} \Delta_c \arg f(z) \]

where \( \Delta_c \) is the increment of \( \arg f(z) \) when we go around \( C \).
Lecture 16, November 4, 85

ROUČE'S THEOREM.

C is a closed contour in the Argand plane. Assume:

1. \( f(z) \), \( g(z) \) are both regular within and on \( C \)
2. \( f(z) \neq 0 \) , \( |g(z)| < |f(z)| \) on \( C \).

Then, \( f(z) \) and \( f(z) + g(z) \) have the same number of zeros within \( C \).

proof: See Copson pp. 119-120.

This theorem is useful to prove the second fundamental theorem of algebra.

EVALUATION OF DEFINITE INTEGRALS

1. \( \int_{0}^{2\pi} R(\cos \theta, \sin \theta) \, d\theta \)

where \( R \) is a rational function.

\[ z = e^{i\theta} \]
\[ \sin \theta = \frac{1}{2i} \left( z - \frac{1}{z} \right) \quad \cos \theta = \frac{1}{2} \left( z + \frac{1}{z} \right) \]
\[ dz = i e^{i\theta} \, d\theta = i \, z \, d\theta \]

\( \Rightarrow \) \( \int_{0}^{2\pi} R(\cos \theta, \sin \theta) \, d\theta = \oint_{C} S(z) \, dz \)

where \( S(z) \) is a rational function of \( z \).
\[ \int_{-\infty}^{\infty} Q(x) \, dx \]

\( Q(x) \) is the boundary value of an analytic function which is regular in the half plane \( \text{Im} z \geq 0 \), except for a finite number of poles, none of them on the real axis.

The so-called **principal value** of the improper integral is defined as:

\[
\int_{-\infty}^{\infty} Q(x) \, dx = \lim_{x_1 \to -\infty} \int_{x_1}^{a} Q(x) \, dx + \lim_{x_2 \to \infty} \int_{a}^{x_2} Q(x) \, dx
\]

which can be shown to be independent of \( a \), provided the two limits exist for a given \( a \).

The **Cauchy principle value** of the integral is defined as:

\[
\lim_{x' \to \infty} \int_{-x'}^{x'} Q(x) \, dx = \text{P} \int_{-\infty}^{\infty} Q(x) \, dx
\]

If the principal value exists, so does the Cauchy principle value. But the opposite is not always true; particularly this happens when there are oscillations of the function which may do the values in \( x_1 \) and \( x_2 \) not to cancel if \( x_1 \neq -x_2 \).
\[
\mathcal{C} \int_{-\infty}^{\infty} P \frac{dx}{x^{4} + \alpha^{4}} = \frac{1}{\alpha^{4}} \left[ 1 + \left( \frac{\alpha}{\beta} \right)^{4} \right] - \frac{1}{\alpha^{4}} \left[ 1 - \left( \frac{\beta}{\alpha} \right)^{4} + \left( \frac{\beta}{\alpha} \right)^{8} - \ldots \right]
\]

\[
\mathcal{C} \int_{-\infty}^{\infty} P \frac{dx}{x^{4} + \alpha^{4}} = \mathcal{O} \left( \frac{1}{\alpha^{4}} \right)
\]
\[ R_1 = \frac{1}{(2\pi^2)(2\pi^2) e^{i(0 + \pi/2 + \pi/4)}} = \frac{e^{-i3\pi/4}}{4\pi^3} \]
\[ = \frac{-1-i}{4\sqrt{2}\pi^3} \]
\[ R_2 = \frac{1}{4\pi^3 e^{i(\pi + 3\pi/4 + \pi/2)}} = \frac{\sqrt{2} e^{-i\pi/4}}{4\sqrt{2}\pi^3} = \frac{1-i}{4\pi^3\sqrt{2}} \]

\[ 2\pi i (R_1 + R_2) = \frac{\pi}{\pi^3\sqrt{2}} \]

The condition \( Q(z) = O\left(\frac{1}{|z|^k}\right), k > 1 \), is equivalent to:
\[ zQ(z) \to 0 \quad \text{as} \quad R \to \infty \]
but this later does not imply the former.

The first condition implies uniform convergence with respect to \( \theta = \arg z \) (0 ≤ \( \arg z \) ≤ π):
\[ |R_1(\theta), C \cdot |Q(z)| \leq \frac{C}{|z|^k} \quad \text{if} \quad |z| \geq R_1(\theta) \]
but \( R_1 \) does not depend on \( \Theta \).

Now let's look closer at the second condition:
Consider
\[ Q(z) = e^{i\theta z} f(z) \]
JORDAN'S INEQUALITY.

If $0 \leq \theta \leq \pi/2$, then, \[ \frac{\theta}{\pi} \leq \frac{\sin \theta}{\theta} \leq 1 \]
interpreted as 1 for $\theta = 0$.

**Proof:**

$\cos \theta$ is decreasing in $[0, \pi/2]$
Therefore, its mean value,
\[ \frac{1}{\pi/2} \int_0^{\pi/2} \cos \theta \, d\theta = \frac{\sin \theta}{\theta} \]
is also decreasing.

$\theta = 0 : \frac{\sin \theta}{\theta} = 1$
$\theta = \pi/2 : \frac{\sin \theta}{\theta} = \frac{2}{\pi}$

JORDAN'S LEMMA.

If:
1. Exists $R_0$, such that $f(z)$ is regular when $\text{Mod}(z) > R_0, 0 \leq \text{arg} z \leq \pi$
2. $f(z) \to 0$, uniformly with respect to $\text{arg} z$, as $|z| \to \infty (0 \leq \text{arg} z)$
3. $m > 0$
Then, \[
\lim_{R \to \infty} \int_{C_R} e^{imz} f(z) \, dz = 0
\]

where \( C_R \) is a semicircle of radius \( R \) above the real axis centered at the origin.

Lecture 17, Nov. 6/85.

Example:
\[
\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2 + a^2} \, dx \quad m > 0 \quad a \text{ real} \neq 0
\]

\( f(z) = \frac{1}{z^2 + a^2} \)

\( f(z) \) is regular for \(|z| > |a|\)

\( f(z) \to 0 \) as \(|z| \to \infty\) uniformly.

Using Jordan's lemma.

\[
I_p \to 0 \quad \text{as} \quad |z| \to \infty \quad 0 \leq \arg z \leq \pi
\]

\[
p \int_{-\infty}^{\infty} \frac{e^{imx}}{x^2 + a^2} \, dx = 2\pi i \sum R
\]

\[
f(z) = \frac{1}{(z + ia)(z - ia)}
\]

\[
R(i\beta) = e^{-m|\beta|}
\]

\[
\Rightarrow \quad p \int_{-\infty}^{\infty} \frac{e^{imx}}{x^2 + a^2} \, dx = \frac{\pi}{|a|} e^{-m|\beta|}
\]
3. \[ \int_0^\infty x^{s-1} f(x) \, dx \quad \text{\(s\) is not an integer} \]

Assume \( f(x) \) is a rational function, \( f(x) \) has no poles on the positive real axis, and \( z^2 f(z) = 0 \) uniformly as \(|z| \to 0\) and also as \(|z| \to \infty\) \((0 \leq \text{arg} \, z < 2\pi)\).

There are branch points at zero and infinity. (for \( z^{s-1} f(z) \))

\[ I = \oint_C z^{s-1} f(z) \, dz \quad \text{where the integrand is taken in the first branch.} \]

\[ z^{s-1} = e^{(s-1) \, \text{log} |z| + i \, \text{arg} \, z} \quad 0 \leq \text{arg} \, z < 2\pi \]

It is clear that \( z^{s-1} f(z) \) doesn't have singularities for \(|z| > R_0\).

(i) Along \( Y \): \( z = \rho e^{i\theta} \)

\[ z^{s-1} f(z) = \rho^{s-1} e^{i\theta(s-1)} f(\rho e^{i\theta}) \]
\[
\int z^{n-1}f(z)\,dz = 8^{n-1}e^{i\theta(n-1)}f(8e^{i\theta})i8e^{i\theta}d\theta = i8^n f(z)\,dz
\]

Since \( z^n f(z) \to 0 \) as \( |z| \to 0 \), uniformly,

\( \exists \varepsilon > 0 \); \( \exists R_0(\varepsilon) \); \( |z^n f(z)| < \varepsilon \) \( \forall |z| > R_0 \)

\( \Rightarrow \left| \int_T z^{n-1}f(z)\,dz \right| \leq \frac{\varepsilon}{\delta} \int_{2\pi R}^{2\pi} = 2\pi \varepsilon \) \( \forall \delta \leq \delta_0 \)

\( \Rightarrow \lim_{\delta \to 0} \int_T z^{n-1}f(z)\,dz = 0 \)

(ii) Along \( T' \): \( z = R e^{i\theta} \)

Since \( z^n f(z) \to 0 \) as \( |z| \to \infty \),

\( \exists \varepsilon > 0 \); \( \exists R_0(\varepsilon) \); \( |z^n f(z)| < \varepsilon \) \( \forall |z| > R_0 \)

\( \Rightarrow \left| \int_T z^{n-1}f(z)\,dz \right| \leq \frac{\varepsilon}{\delta} \int_{2\pi R}^{2\pi} = 2\pi \varepsilon \) \( \forall R > R_0 \)

(iii) \( L^+ \): \( \int_L z^{n-1}f(z)\,dz = \int_{z=8}^{R} x^{n-1}f(x)\,dx \)

(iv) \( L^- \): \( \int_L z^{n-1}f(z)\,dz = \int_{(8e^{2\pi i})^{n-1}}^{R} f(x)\,dx \)
\[
= \int_{\gamma} x^{\alpha-1} e^{2\pi i \alpha (z-1)} f(x) \, dx \\
= -e^{2\pi i \alpha (z-1)} \int_{\gamma} x^{\alpha-1} f(x) \, dx = -e^{2\pi i \alpha} \int_{\gamma} x^{\alpha-1} f(x) \, dx
\]

Adding the four parts, and taking the limit \( s \to 0, R \to \infty, \)

\[
(1 - e^{2\pi i \alpha}) \int_{\gamma} x^{\alpha-1} f(x) \, dx = 2\pi i \sum R
\]

\[
\int_{\gamma} x^{\alpha-1} f(x) \, dx = -\pi e^{-i\pi \alpha} \csc(\pi \alpha) \sum R
\]

Example:

\[
\int_{\gamma} \frac{x^{\alpha-1}}{1+x} \, dx, \quad 0 < \alpha < 1
\]

\[
f(z) = \frac{1}{1+z} \text{ has no poles on the positive real axis.}
\]

\[
\frac{2^\alpha}{1+z} \sim z^{\alpha} \to 0, \text{ if } |z| \to 0, \text{ if } \alpha > 0
\]

\[
\frac{2^\alpha}{1+z} \sim z^{\alpha-1} \to 0, \text{ as } |z| \to \infty, \text{ if } \alpha < 1
\]

\[
\Rightarrow \int_{\gamma} \frac{x^{\alpha-1}}{1+x} \, dx = -\pi e^{-i\pi \alpha} \csc(\pi \alpha) \sum R = -\pi e^{-i\pi \alpha} \csc(\pi \alpha) \sum (-1)^{\alpha-1}
\]

The only singularity is a simple pole at \( z = -1 \)
Hints for Some other types of integrals:

Lecture 18, Nov. 11/85

**COMPLEX TRANSFORMATION.**

\[
\int_0^\infty e^{-x^2} \cos 2bx \, dx \quad , \quad b > 0
\]

Consider:

\[
\oint e^{-z^2} \, dz = 0
\]

1. \[
\int_0^\infty e^{-z^2} \, dz \to \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} \quad (as \quad x \to \infty)
\]

2. \[
\int_0^{a+ib} e^{-z^2} \, dz = i \int_0^b e^{-(x+iy)^2} \, dy = ie^{-a^2} \int_0^b e^{y^2-2iay} \, dy
\]

\[
\left| \int_0^{a+ib} e^{-z^2} \, dz \right| \leq e^{-a^2} M b \quad \text{where} \quad M = \max e^{y^2} = e^{b^2}
\]

\[
\int_{a+ib}^{b+i0} e^{-z^2} \, dz \to 0 \quad , \quad as \quad a \to \infty
\]
\[ \int_{\text{ib}}^{\text{a}+\text{ib}} e^{-x^2} \, dx = \int_{0}^{\infty} e^{-(x+ib)^2} \, dx = e^{b^2} \int_{0}^{\infty} e^{-x^2} (\sin 2bx - \cos 2bx) \, dx \]

\[ \rightarrow e^{b^2} \int_{0}^{\infty} e^{-x^2} (\sin 2bx - \cos 2bx) \, dx \]

\[ \int_{\text{ib}}^{\text{b}} e^{-x^2} \, dy = -i \int_{0}^{b} e^{y^2} \, dy \]

\[ \Rightarrow \frac{\sqrt{\pi}}{2} + e^{b^2} \int_{0}^{\infty} e^{-x^2} (-e^{-2ixb}) \, dx - i \int_{0}^{b} e^{y^2} \, dy = 0 \]

\[ \Rightarrow \left\{ \begin{array}{l}
\int_{0}^{\infty} e^{-x^2} \cos(2bx) \, dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \\
\int_{0}^{\infty} e^{-x^2} \sin(2bx) \, dx = e^{-b^2} \int_{0}^{b} e^{y^2} \, dy
\end{array} \right. \]

"Intuition is condensed experience."

**FRESNEL INTEGRALS.**

\[ \int_{0}^{\infty} \cos^2 t \, dt, \int_{0}^{\infty} \sin^2 t \, dt \]

---

(1) Wolf's advisor (I guess that is Max Born)
$$\oint_C e^{-z^2} \, dz = 0$$

$$\int_A^B e^{-z^2} \, dz = \left. iR \right|_{\frac{\pi}{2}}^{\frac{\pi}{4}} e^{-R^2 (\cos \phi + i \sin \phi)} e^{i \phi/2} \, d\phi = I_2$$

$$= iR \left. \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{-R^2 (\cos \phi + i \sin \phi)} e^{i \phi/2} \, d\phi \right|$$

$$|I_2| \leq \frac{R}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{-R^2 \cos \phi} \, d\phi = \frac{R}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{-R^2 \sin \phi} \, d\phi$$

By means of Jordan's inequality,

$$-R^2 \sin \phi \leq -\frac{2R^2}{\pi} \phi$$

$$|I_2| \leq \frac{R}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{-\frac{2R^2}{\pi} \phi} \, d\phi = \frac{R}{2} \left( -\frac{\pi}{2R^2} \right) (e^{-R^2} - 1)$$

$$|I_2| \to 0 \quad \text{as} \quad R \to \infty$$

$$\zeta = t e^{i \pi/4} = \frac{1+i}{\sqrt{2}} t \quad \zeta^2 = it^2$$

$$\int_B^A e^{-z^2} \, dz = \int_0^R e^{-it^2} \left( \frac{1+i}{\sqrt{2}} \right) dt = -\frac{1+i}{\sqrt{2}} \int_0^R e^{-it^2} \, dt$$

$$\Rightarrow \quad \frac{i\pi}{2} = \frac{1+i}{\sqrt{2}} \int_0^\infty e^{-it^2} \, dt$$

$$\int_0^\infty e^{-it^2} \, dt$$

$$\int_0^\infty \cos t^2 \, dt - i \int_0^\infty \sin t^2 \, dt = \frac{i\pi}{2} e^{-i\pi/4}$$
\[
\begin{align*}
\int_0^\infty \cos^2 t \, dt &= \frac{\sqrt{2\pi}}{4} \\
\int_0^\infty \sin^2 t \, dt &= \frac{\sqrt{2\pi}}{4}
\end{align*}
\]

**METHOD OF PARAMETRIC DIFFERENTIATION.**

\[
I = \int_{-\infty}^{\infty} \frac{x e^{imx}}{x^2 + a^2} \, dx \quad (m > 0, a \text{ real})
\]

\[
I = \int_{-\infty}^{\infty} \frac{2}{2m} \left\{ \frac{1}{i} \frac{e^{imx}}{x^2 + a^2} \right\} \, dx = \frac{1}{2m} \int_{-\infty}^{\infty} \frac{e^{imx}}{x^2 + a^2} \, dx
\]

we have already evaluated the last integral:

\[
I = \frac{1}{i} \frac{2}{2m} \left( \frac{\pi}{1a} e^{-m|a|} \right) = -\pi \frac{e^{-m|a|}}{a}
\]

\[
\frac{\partial I}{\partial m} = i \int_{-\infty}^{\infty} \frac{x^2 e^{imx}}{x^2 + a^2} \, dx
\]

\[
\frac{e^2}{e^2 + a^2} \sim 1 \text{ and therefore the integral diverges;}
\]

besides this problem, this method also has the inconvenience that sometimes the integration and differentiation may not be exchanged; this is possible if:

1. The integrand is continuous.
2. The integral converges uniformly.

*(See Copson, pp.108-110)*
\[ \int \frac{x^n}{(x^2+a^2)^m} \, dx \quad \int \frac{x^n e^{imx}}{(x^2+a^2)^m} \, dx \]

The first integral may be obtained from the second one, as the limit when \( m \to \infty \), provided the limit and integration can be commuted.

Copson, pp. 106-110

**IMPROPER INTEGRALS.**

\[ \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx = P \int_{-\infty}^{\infty} f(x) \, dx \]

If \( f(x) \) has a singularity in a point \( x_0 \) at the real axis, the integral is called improper.

\[ \lim_{\epsilon_1 \to 0} \int_{0}^{x_0-\epsilon_1} f(x) \, dx + \lim_{\epsilon_2 \to 0} \int_{x_0+\epsilon_2}^{b} f(x) \, dx \]

If this two limits exist, their sum is called the value of the improper integral:

\[ \int_{0}^{b} f(x) \, dx \]
The Cauchy principle value of the integral is defined as:

\[ \lim_{\varepsilon \to 0} \left\{ \int_{x_0 - \varepsilon}^{x_0} f(x) \, dx + \int_{x_0 + \varepsilon}^{b} f(x) \, dx \right\} = P \int_{a}^{b} f(x) \, dx \]

**Examples**

1. \[ \int_{0}^{1} \frac{1}{\sqrt{x}} \, dx = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \frac{1}{\sqrt{x}} \, dx = 2 \sqrt{\varepsilon} \bigg|_{\varepsilon}^{1} \to 2 \]

2. \[ \int_{-1}^{1} \frac{1}{x} \, dx = \lim_{\varepsilon_{1} \to 0} \int_{-\varepsilon_{1}}^{\varepsilon_{1}} \frac{1}{x} \, dx + \int_{\varepsilon_{2}}^{1} \frac{1}{x} \, dx \quad \text{(with } \varepsilon_{1} \to 0) \]

\[ = \int_{\varepsilon_{1}}^{\varepsilon_{2}} \frac{1}{x} \, dx \quad \text{(with } \varepsilon_{1} \to 0) \]

Both integrals diverge, but the principle value does not:

\[ P \int_{-1}^{1} \frac{1}{x} \, dx = \lim_{\varepsilon \to 0} \int_{-1}^{\varepsilon} \frac{1}{x} \, dx + \int_{\varepsilon}^{1} \frac{1}{x} \, dx = \lim_{\varepsilon \to 0} (\ln \varepsilon - \ln \varepsilon) = 0 \]

**PRINCIPLE VALUE OF AN INTEGRAL OF COMPLEX VARIABLE.**
CAUCHY'S RESIDUE THEOREM EXTENDED

Suppose \( f(z) \) analytic and single-valued inside and on the boundary of a domain \( D \). (Actually, \( f(z) \) must be analytic also in a region very close to the boundary outside \( D \).) If \( C \) is a smooth curve. (Where \( C \) is the boundary of \( D \).)

\[
\oint_{C} f(z) \, dz + \oint_{\Gamma} f(z) \, dz = 2\pi i \sum R
\]

Where the residues in the last sum are only those at points singular inside \( D \).

Then,

\[
P \oint_{C} f(z) \, dz + \lim_{\varepsilon \to 0} \oint_{\Gamma} f(z) \, dz = 2\pi i \sum R
\]
Assume that $z_0$ is a simple pole for $|z - z_0| \leq \epsilon$,

$$f(z) = \frac{B}{z - z_0} + g(z)$$

regular

$$z = z_0 + \epsilon e^{i\theta}$$

$$\oint_{\gamma} f(z)\,dz = -\int_{\gamma} \left[ \frac{B}{E e^{i\theta}} + g(z_0 + \epsilon e^{i\theta}) \right] i \epsilon e^{i\theta} \,d\theta$$

$$= -iB \int_{\pi}^{0} d\theta = i \epsilon \int_{\pi}^{0} g(z_0 + \epsilon e^{i\theta}) e^{i\theta} \,d\theta \rightarrow -i\pi R(z_0)$$

$$\Rightarrow \quad P \oint_{C} f(z)\,dz = 2\pi i \left( \sum R + \frac{1}{2} \sum_{i=0}^{n} R(z_i) \right)$$

In general, if $z_0, z_1, \ldots, z_n$ are the singularities of $f(z)$ on the boundary,

$$P \oint_{C} f(z)\,dz = 2\pi i \left( \sum R + \frac{1}{2} \sum_{i=0}^{n} R(z_i) \right)$$

where $\sum R$ is the sum of residues of $f(z)$ inside $D$.

The same result is obtained if we choose $\gamma$ going outside $D$. 
\[
\int_C f(z) \, dz - \int_C f(z) \, dz = 2\pi i \, R(z_0)
\]
\[
\int_C f(z) \, dz + \int_C f(z) \, dz = 2\pi i \left(\sum R + R(z_0)\right)
\]
\[
\Rightarrow \quad P \int_{f(z)} \, dz + \pi i \, R(z_0) = 2\pi i \left(\sum R + R(z_0)\right)
\]
\[
P \int_{f(z)} \, dz = 2\pi i \left(\sum R + \frac{1}{2} R(z_0)\right)
\]

In some cases when the contour is not smooth at \( z_0 \), we still can find a similar result:

\[
\lim_{\epsilon \to 0} \int_{\gamma} f(z) \, dz = i \, R(z_0) \omega
\]

where \( \omega \) is the angle between the two tangents at \( z_0 \).

**Cauchy's (Second) Integral for \( f(z) \), when \( \omega \) is on the contour.**

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - \omega} \, dz = \begin{cases} 
    f(\omega), & \omega \text{ is within } C \\
    0, & \omega \text{ is outside } C
\end{cases}
\]
what happens when \( a \) is right on the contour \( C \)?

\[ F(z) = \frac{f(z)}{z-a} \]

is regular inside \( C \), and has a simple pole of residue \( f(a) \) at the point \( z=a \).

Therefore,

\[ \text{P \quad \oint_C \frac{f(z)}{z-a} \, dz = 2\pi i \left( \frac{1}{2} f(a) \right) } \]

\[ \Rightarrow \quad \frac{1}{2\pi i} \text{P \quad \oint_C \frac{f(z)}{z-a} \, dz} = \frac{1}{2} f(a) \quad \text{if } a \text{ is on } C \]

\[ \frac{1}{2\pi i} \text{P \quad \oint_C \frac{f(z)}{z-a} \, dz} = \begin{cases} f(a), & a \text{ inside } C \\ \frac{1}{2} f(a), & a \text{ on } C. \\ 0, & a \text{ outside } C. \end{cases} \]

Lecture 20, Nov. 18/85

Conjugate functions and solutions of the Dirichlet problem for a half plane and a circle.

\[
\begin{align*}
\text{Dirichlet problem: A solution of Laplace's equation is wanted,} \\
\text{with a fixed value } u(x,y) \text{ on the boundary } C.
\end{align*}
\]
Another problem of interest is to find $u(x,y)$ such that $\nabla^2 u$ inside $C$, and $u(x,y)$ is known on $C$.

The values of $u(x,y)$ and $v(x,y)$ at the boundary are related.

**1. Half-Plane**

$f(z)$ is regular in the upper half-plane, and decreases sufficiently rapidly as $|z| \to \infty$.

$$f(z) = \frac{1}{2\pi i} \left[ \int_{-R}^{R} \frac{f(x)}{x-z} \, dx + \int_{\gamma} \frac{f(z)}{z-z^*} \, dz \right] \quad \text{(if $R$ is large enough)}$$

$$\int_{-R}^{R} \frac{f(x)}{x-z} \, dx + \int_{\gamma} \frac{f(z)}{z-z^*} \, dz = 0$$

Since $f(z)$ decreases sufficiently rapidly,

$$\lim_{R \to \infty} \int_{\gamma} \frac{f(z)}{z-z^*} \, dz \to 0 \quad \lim_{R \to \infty} \int_{\gamma} \frac{f(z)}{z-z^*} \, dz \to 0$$

Therefore what we mean here by sufficiently rapidly is:

$$f(z) = O\left(\frac{1}{|z|}\right)$$
we also assume that the two integrals along the x-axis exist in the ordinary sense, as \( R \to \infty \).

\[
\left\{ \begin{array}{l}
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x-a} \, dx = f(a) \\
\int_{-\infty}^{\infty} \frac{f(x)}{x-a^*} \, dx = 0
\end{array} \right.
\]

\[\Rightarrow \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(x) \left[ \frac{1}{x-a} \pm \frac{1}{x-a^*} \right] \, dx = f(a)\]

\[
\frac{1}{x-a} \pm \frac{1}{x-a^*} = \left\{ \begin{array}{l}
\frac{2(x-\xi)}{(x-\xi)^2 + \eta^2}, + \\
-\frac{2(i\eta)}{(x-\xi)^2 + \eta^2}, -
\end{array} \right.
\]

\[\Rightarrow \left\{ \begin{array}{l}
\frac{\eta}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{(x-\xi)^2 + \eta^2} \, dx = f(\xi + i\eta) \\
\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{f(x)(x-\xi)}{(x-\xi)^2 + \eta^2} \, dx = \Phi(\xi + i\eta)
\end{array} \right.\]

If \( f(x+iy) = u(x,y) + i \, v(x,y) \), then

\[
\begin{align*}
\quad \quad u(\xi, \eta) &= \frac{\eta}{\pi} \int_{-\infty}^{\infty} \frac{u(x,0)}{(x-\xi)^2 + \eta^2} \, dx \\
v(\xi, \eta) &= \frac{\eta}{\pi} \int_{-\infty}^{\infty} \frac{v(x,0)}{(x-\xi)^2 + \eta^2} \, dx
\end{align*}
\quad \quad \eta > 0
\]
The second equation gives:

\[
U(\frac{x}{\xi}, n) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-\xi)u(x,0)}{(x-\xi)^2 + n^2} \, dx \quad n > 0
\]

\[
\mathcal{U}(\frac{x}{\xi}, n) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-\xi)u(x,0)}{(x-\xi)^2 + n^2} \, dx
\]

if \( n=0 \), an additional factor of \( \frac{1}{n} \) must be included on Cauchy's equation, and the integrals must be replaced by their principle values.

\[
U(\frac{x}{\xi}, 0) = \frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} \frac{u(x,0)}{x-\xi} \, dx
\]

\[
\mathcal{U}(\frac{x}{\xi}, 0) = -\frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} \frac{u(x,0)}{x-\xi} \, dx
\]

**Titchmarsh's Theorem.**

(Ref: Titchmarsh, Introduction to the theory of Fourier integrals, Oxford, 1948)

If \( u(x) \) and \( \mathcal{U}(x) \) are two real square integrable functions of the real variable \( x \), which are connected by the Hilbert transform, then...
\[ u(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(x')}{x'-x} \, dx' \quad \text{and} \quad v(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(x')}{x'-x} \, dx' \]

then \( f(x) = u(x) + iv(x) \) is the limit as \( y \to 0 \) of an analytic function \( f(x+iy) \) which is regular for \( y > 0 \), and,

\[ \int_{-\infty}^{\infty} |f(x+iy)|^2 \, dx \leq K \quad \text{for} \quad y > 0 \]

where \( K \) is a constant


**INSIDE A CIRCLE**

If \( a \) is inside \( C \),

\[ \frac{1}{2\pi i} \oint_{C} \frac{f(z)}{z-a} \, dz = f(a) \]

The "image" of \( a \) is a point \( b \) in the same line through \( 0 \), and such that \( \text{dist}(0,a) \leq \text{dist}(0,b) = R^2 \).

\[ a = re^{i\theta} \]
\[ b = R^2 e^{i\theta} \]

\[ |ab| = r \cdot R = R^2 \]

\[ \boxed{b = \frac{R^2}{r^*}} \]

an alternate definition of the inverse point.
By means of Cauchy integral equation around \( a \) and \( b \), we get,

\[
\begin{align*}
U(r, \theta) &= U(0) + \frac{Rr}{\pi} \int_0^{2\pi} \frac{\sin(\phi - \theta)}{R^2 - 2Rr \cos(\phi - \theta) + r^2} U(R, \phi) \, d\phi \\
V(r, \theta) &= V(0) - \frac{Rr}{\pi} \int_0^{2\pi} \frac{\sin(\phi - \theta)}{R^2 - 2Rr \cos(\phi - \theta) + r^2} U(R, \phi) \, d\phi
\end{align*}
\]

**Also:**

\[
\begin{align*}
U(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos(\phi - \theta) + r^2} U(R, \phi) \, d\phi \\
V(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos(\phi - \theta) + r^2} V(R, \phi) \, d\phi 
\end{align*}
\]

\( r < R \)

\( U(0) \) and \( V(0) \) are two arbitrary constants that come from the solution of Cauchy-Riemann equations. In the case of the half plane, they didn't appear because we assumed that \( f(z) \to 0 \) as \( |z| \to \infty \). (boundary conditions); in this cases \( U(0) \) and \( V(0) \) can also be determined from the boundary conditions.
\[ U(R, \phi) = U(0) + \frac{1}{2\pi} \int_0^{2\pi} U(R, \phi) \cot \left( \frac{\phi - \phi_0}{2} \right) d\phi \]
\[ V(R, \phi) = V(0) - \frac{1}{2\pi} \int_0^{2\pi} U(R, \phi) \cot \left( \frac{\phi - \phi_0}{2} \right) d\phi \]

These equations are analogous to Hilbert transforms, and they allow us to find \( U(R, \phi) \) knowing \( V(R, \phi) \) and vice versa.

**Conformal Mapping.**

If \( \alpha = \alpha' \) for every pair of curves \( C_1, C_2 \), the mapping is said to be **isogonal**.

Furthermore, if the sense of \( \alpha \) is preserved, the mapping is said to be **conformal**.

This concept of conformal mapping is not restricted to mappings from the plane to the plane; in the case of a conformal mapping from the sphere to the plane, it is called **stereographic (or Mercator's) projection**.
Riemann (1851), Gauss (1822).

**THEOREM.**
Any domain on the plane (except the whole plane itself) can be conformally mapped into the unitary circle.

**THEOREM**
Every regular, analytic function on D, generates an isogonal mapping of D. (except if \( f(z) = 0 \)).

Copson pp. 121, 123

If \( f(z) \) is regular, \( \alpha = \Delta \).

**Proof:**
\[
\begin{align*}
Z - z_0 &= r_1 e^{i\Theta_1} \\
Z_2 - z_0 &= r_2 e^{i\Theta_2} \\
W - w_0 &= r_1' e^{i\Theta_1'} \\
W_2 - w_0 &= r_2' e^{i\Theta_2'}
\end{align*}
\]

if \( r_1, r_2 \to 0 \), \( \Theta_1 - \Theta_2 \to \Delta \).

suppose \( r_1', r_2', \Theta_1', \Theta_2' \) are continuous functions of \( r_1, r_2, \Theta_1, \Theta_2 \).
\[ r_1 \to 0 \Rightarrow r_1' \to 0 \Rightarrow \frac{r_1'}{r_1} \to \text{definite value} \]

The derivative of \( f(z) \) at \( z_0 \), evaluated along \( C \), is:

\[ f'(z_0) = \lim_{z \to z_0} \frac{W_1 - W_0}{Z_1 - Z_0} = \lim_{z \to z_0} \frac{r_1' e^{i\theta_1}}{r_1 e^{i\theta_0}} = Re^{i\delta} \]

(if \( f'(z_0) \neq 0 \))

where \( R = \lim_{z \to z_0} \frac{r_1'}{r_1} \)

\[ \delta = \lim_{z \to z_0} (\theta_1' - \theta_1) = \angle_2' - \angle_1' \]

along \( C_2 \), \( f'(z_0) \to Re^{i\delta} \)

\[ R = \lim_{z \to z_0} \frac{r_2'}{r_2} \]

\[ \delta = \lim_{z \to z_0} (\theta_2' - \theta_2) = \angle_2' - \angle_2 \]

If \( f(z) \) is regular, and \( f'(z_0) \neq 0 \), then,

\[ \angle_2' - \angle_2 = \angle_1' - \angle_1 \]

\[ \angle_2' - \angle_1' = \angle_2 - \angle_1 \]

\[ \Rightarrow \angle_2' = \angle_2 \]

and the mapping is isogonal.

Also, if \( r_1, r_2 \) are small,

\[ \frac{r_1'}{r_2'} \approx \frac{r_1}{r_2} = R \Rightarrow \]

which means that a small geometric figure
is transformed into a figure geometrically similar to it.

The figure in the \( w \)-plane is the same as the one in the \( z \)-plane, magnified by a factor \( R = |f'(z_0)| \) and rotated at an angle \( S = \text{arg}(f'(z_0)) \).

**CRITICAL POINTS.**

The points where \( f'(z) = 0 \) are called singular points of the conformal mapping generated by \( f(z) \).

**THEOREMS**

If \( u(x,y) \), \( v(x,y) \) are analytic and regular in a domain \( D \). \( u_x, u_y, v_x, v_y \) are continuous and do not vanish simultaneously at any point of \( D \), and \( u \) and \( v \) are single valued.

The ratio of corresponding linear elements in the mapping that they establish is independent on the direction.

Then, the mapping is isogonal and

\[
w = f(z) = u + i v \quad \text{or} \quad w = f(z) = u - i v
\]
where \( f(z) \) is analytic function, regular in \( D \).

We have already proved that \( w = u + iv \) and \( w = u - iv \) are isogonal mappings. The case \( u + iv \) gives also a conformal mapping, and \( u - iv \) inverts the sense of the angle.

**RIEMANN'S BASIC THEOREM ON CONFORMAL MAPPING.** (1851)

A Jordan curve is continuous and with no multiple points. If \( D \) is any open domain bounded by a closed Jordan curve \( \gamma \), then there is a function \( f(z) \), analytic and regular in \( D \), such that \( w = f(z) \) maps \( D \) conformally into the interior of the circle \( |w| = 1 \).

There are several such functions \( f(z) \); however there is one amongst them which will also transform \( z_0 \) into the origin, and a given direction through \( z_0 \) into the direction of the real axis.

\( (*) \) open means that the boundary is not included; but still \( D \) is bounded.
Poisson's formula. It gives the only bounded solution even in the case when $u(R, \phi)$ is discontinuous and the solution is not unique.

This theorem is equivalent to the existence and uniqueness of the solution of Laplace's equation, with Dirichlet boundary conditions.

Riemann proved it in 1851, but Weierstrass found that his proof failed.
Hilbert proved it in 1901.
(See Kellog p. 279, ...)

Sketch of the proof:

Moreover, $z_0$ is a simple zero of $f(z)$. 
\[ f(z) = f(z_0) + (z-z_0)f'(z_0) + \ldots \]

\[ \frac{f(z)}{z-z_0} \text{ is well behaved throughout } D, \text{ including } z_0. \]

\[ \Rightarrow f(z) = (z-z_0)f'(z_0) + \ldots \text{ converges for } \forall z \in D, z \neq z_0 \]

\[ \frac{f(z)}{z-z_0} \text{ can not vanish anywhere in the domain } D : \]

\[ \text{if } \frac{f(z)}{z-z_0} = 0 \Rightarrow \begin{cases} f(z_0) = 0 & \text{which is not true,} \\ f(z_1) = 0 & \text{since } f(z) \text{ is 1to1}. \end{cases} \]

\[ e^{\gamma(z)} = \frac{f(z)}{z-z_0} \]

\[ \gamma(z) = \log \left( \frac{f(z)}{z-z_0} \right) \text{ is analytic and regular in } D. \]

Let \[ \gamma(z) = \phi(z) + i \chi(z) \], where \( \phi, \chi \) are real.

\[ \Rightarrow |e^{\gamma(z)}| = e^{\phi(z)} = \frac{|f(z)|}{|z-z_0|} \]

\[ |f(z)| = 1, \forall z \in C. \]

\[ \Rightarrow \phi(z) = \log \frac{|f(z)|}{|z-z_0|} = -\log|z-z_0|, \forall z \in C. \]

Then, we have the value of the harmonic function \( \phi(z) \) on the boundary, and the problem of finding \( f(z) \) has reduced to the Dirichlet problem:

\[ \begin{cases} \nabla^2 \phi(z) = 0, \forall z \in D \\ \phi(z) = -\log|z-z_0|, \forall z \in C. \end{cases} \]
Or, equivalently,

\[
\begin{align*}
\nabla^2 V(x,y) &= 0 \quad (x,y) \in D, (x,y) \neq (0,0) \\
V(x,y) &= 0 \quad (x,y) \in C
\end{align*}
\]

where \( V(x,y) = \log |z-z_0| + \phi(z) \).

The behaviour of \( V \) at the point \( z_0 \) is the same as \( \log |z-z_0| \).

\[\text{\scriptsize{\textbullet\, line of charge.}}\]

Intuitively (for the physicist) there is a unique solution to this problem.

After we have found \( V(x,y) \), we have an arbitrary constant when we determine \( \chi(x,y) \), which determines specifying the direction through \( z_0 \), which we want to map through the \( x \) axis.

\[ f(z) = (z-z_0) e^{[\phi(z) + i \chi(z) \text{ const.}]}. \]

O.D. Kellogg. \textit{Foundations of potential theory}\textit{.}
SOLUTION OF DIRICHLET PROBLEM BY MEANS OF CONFORMAL MAPPING.

\[ \nabla^2 U(x, y) = 0, \quad (x, y) \in \Omega \]
\[ U(x, y) = g(x, y), \quad (x, y) \in \Gamma \]

\[ \nabla^2 V(x, y) = 0, \quad \Omega^* \]
\[ V(x, y) = g^*(x, y), \quad \Gamma \]

where \( u(x, y) + iv(x, y) = f(z) = f(x + iy) \).

Choosing an appropriate mapping, the problem may be reduced to a simpler one.

BILINEAR MAPPING. (Moebius mapping, homography).

\[ w = \frac{az + b}{cz + d} \quad a, b, c, d \text{ constants} \]
\[ ad - bc \neq 0 \]

It maps the whole plane into the whole plane, with the exception of a finite number of points (including the point at infinity).

(Copson, p. 187-188).

\[ z = \frac{dw - b}{-cw + a} \]

is also a bilinear mapping, and is the inverse of the first one.

It is obvious that the two mappings are conformal.

\[
\begin{array}{c|c|c}
\infty & W & \infty \\
\hline
-\frac{a}{c} & \infty & \frac{a}{c}
\end{array}
\]
if \( c = 0 \), the points at infinity on the \( z \) and \( w \) planes, correspond to each other.

\[
W = \frac{z}{c} + \frac{z_3}{c}
\]

\[
z_3 = \frac{bc - ad}{c^2} \cdot z_2
\]

\[
z_2 = \frac{1}{z_1}
\]

\[
z_1 = z + \frac{d}{c}
\]

This four transformations are simple transformations:

(i) \( z_1 = z - \frac{d}{c} \) \text{ translation.} \]

(ii) \( z_1 = r, e^{i\theta}, z_2 = \frac{1}{r}, e^{-i\theta} \) \text{ "Imagining" through the unit circle and reflection about the x-axis} \]

(iii) \( z_3 = \frac{bc - ad}{c^2} z_2 \) \text{ is a magnification (change of scale) and a rotation.} \]

(iv) \( W = \frac{z}{c} + z_3 \) \text{ is a translation.} \]

\[
W = \text{translation inversion reflection magnification rotation.}
\]
Lecture 23, 12/2.

A bilinear mapping maps circles into circles, and pairs of inverse points into pairs of inverse points.

SPECIAL CASE OF BILINEAR MAPPING.
Suppose a bilinear mapping such that:

\[ \begin{align*}
Z_0 &\rightarrow W = 0 \\
Z_\infty &\rightarrow W = \infty \\
Z = 0 &\rightarrow \overline{W_0}
\end{align*} \]

Then, it has to be of the form:

\[ W = A \frac{Z - Z_0}{Z - Z_\infty} \]

\[ W_0 = A \frac{Z_0}{Z_\infty} \implies A = \frac{W_0 Z_\infty}{Z_0} \]

\[ W = (W_0 \frac{Z_\infty}{Z_0}) \left( \frac{Z - Z_0}{Z - Z_\infty} \right) \]

We are interested now in the mapping

\[ \text{Im}(Z) \rightarrow |W| < 1 \]
\[ W = \frac{az + b}{cz + d} \]
\[ z_0 \rightarrow w = 0 \quad \Rightarrow \quad az_0 + b = 0 \]

The image of \( z_0 \) (or its inverse) on the \( z \)-plane is the origin; and the inverse of the origin on the circle is \( w = \infty \), then,
\[ z_\infty = z_0^* \rightarrow w = \infty \]
\[ cz_\infty + d = 0 \]
\[ z_\infty = -\frac{b}{c} \quad , \quad z_0^* = -\frac{d}{c} \]

Therefore, two of the constants \( a, b, c, \) and \( d \) can be eliminated.

\[ W = \left( \frac{2}{c} \right) \frac{z - z_0}{z - z_0^*} \]

Another constant can be eliminated choosing a direction which goes through the real axis in the \( w \)-plane; also we must choose the way the origin is mapped:
\[ z = 0 \rightarrow |W| = 1 \]
\[ \left| \frac{2}{c} \right| \left| \frac{z_0}{z_0^*} \right| = 1 \quad \Rightarrow \quad \left| \frac{2}{c} \right| = 1 \]

\[ W = e^{i\theta} \frac{z - z_0}{z - z_0^*} \]

depends on the direction we choose.
\[ S = \arg f'(z) \bigg|_{z = z_0} \]

\[ \Rightarrow S = \theta - \pi/2 \]

For \( S = 0 \) (x direction going into x direction), \( \theta = \pi/2 \)

\[ W = i \frac{z - z_0}{z - \overline{z}_0} \]

Another interesting mapping is one which transforms the interior of the unit circle into itself:

\[ W = e^{i\theta} \frac{z - z_0}{1 - \overline{z}_0 z} \]

Ref:
- Copson, p. 190
- Joukowski. As a final example, consider the mapping
  \[ W = z^n \quad n \text{ integer} > 1 \]

W is an entire function, but the mapping has a singularity at \( z = 0 \):

\[ W' \big|_{z = 0} = 0 \]

\[ z = re^{i\theta}, \quad W = se^{i\phi} \]

\[ \Rightarrow \begin{cases} s = r^n \\ \phi = n\theta \end{cases} \]
Circle around the origin → circle around the origin, magnified n times

Angles are not preserved: \( \theta \rightarrow n\theta \)

The inverse transform,
\[
z = \sqrt[n]{w}
\]

assigns \( n \) points \( z \) to a point in the \( w \)-plane.

\( 0 < \theta < 2\pi/n \)

Morse & Feshbach.
ORDINARY LINEAR DIFFERENTIAL EQUATIONS.

A partial differential equation of \( n \) variables can be replaced by \( n \) ordinary differential equations.

Ref.

Capon, first part, Ch. X.

Morse & Feshback, VI, § 5.2


\[
L[y(z)] = F(z)
\]

where, \( L = \frac{d^n}{dz^n} + p_1(z) \frac{d^{n-1}}{dz^{n-1}} + \ldots + p_n(z) \)

and \( F(z), p_1(z), \ldots, p_n(z) \) are complex functions.

\( n \) is called the ORDER of the equation.

If \( F(z) = 0 \), the equation is called HOMOGENEOUS.

For a homogeneous equation, it is easy to prove that if \( y_1(z), \ldots, y_n(z) \) are solutions, then

\[
\sum_{k=1}^{N} c_k y_k(z), \quad c_k = \text{constants}
\]

is also a solution.
For inhomogeneous equations, if
\[ L[\gamma_0^i(z)] = F(z) \]
and \[ L[\gamma^k(z)] = 0 \]
then,
\[ L[\gamma_0^i(z) + \gamma^k(z)] = F(z) \]
\[ \gamma = \gamma_0^i + \gamma^k \] is a solution of the equation.
Moreover, it can be proved that if \( \gamma_0^i(z) \) is the most general homogeneous equation, \( \gamma \) is the most general sol. of the inhomogeneous equation.

Proof.
\[ \gamma(z) = \gamma_0^i(z) + \chi(z) \]
\[ L[\gamma(z)] = L[\gamma_0^i(z)] + L[\chi(z)] = F(z) + L[\chi(z)] = F(z) \]
\[ \Rightarrow L[\chi(z)] = 0 \]

Let \( \gamma = \chi_1, \gamma^{(1)} = \chi_2, \gamma^{(2)} = \chi_3, \ldots, \gamma^{(m-1)} = \chi_m \)

where \( \gamma^{(k)} = \frac{d^k \gamma}{dz^k} \)
and $\psi(z)$ is solution of

$$\frac{d^n}{dz^n} \psi(z) + p_1 \frac{d^{n-1}}{dz^{n-1}} \psi(z) + \ldots + p_n \psi(z) = 0$$

\[ \Rightarrow \frac{dX_n}{dz} = \frac{d^n \psi}{dz^n} = -p_1 X_n - p_2 X_{n-1} - \ldots - p_n X_1 \]

doing the same for $n=1, 2, \ldots, n$, we get $n$ ordinary differential equations of first order.

Lecture 24, Dec. 4/95

Suppose $\{p_n(z)\}$ are analytic and regular within a circle $C$ centered on $z=z_0$.

Then, there is a unique set of solutions, $X_1, X_2, \ldots, X_n$, analytic and regular within $C$, and such that,

$$X_k(z_0) = \lambda_k$$

This allows us to find a unique solution $\psi(z)$, with the constraints:

$$\psi(z_0) = \lambda_1, \quad \frac{\partial \psi}{\partial z} \bigg|_{z=z_0} = \lambda_2, \quad \ldots, \quad \frac{d^{n-1} \psi}{dz^{n-1}} \bigg|_{z=z_0} = \lambda_n$$

This theorem is FUCH'S THEOREM.

ORDINARY POINTS AND SINGULAR POINTS.

$z = z_0$ is a regular point of the differential equation, if $P_j(z_0)$ is regular at $z_0$, $j=1, \ldots, n$. Otherwise, $z_0$ is called a singular point.

SECOND ORDER EQUATIONS.

$$\frac{d^2 \varphi(z)}{dz^2} + p(z) \varphi(z) = 0 \quad \text{(first order)}$$

$$\frac{d}{dz} (\log \varphi) = -p(z)$$

$$\Rightarrow \log \varphi(z) = -\int_{z_0}^{z} p(z) \, dz$$

$$\varphi(z) = \varphi(z_0) e^{-\int_{z_0}^{z} p(z) \, dz}$$

For more general equations it is not possible to write down a general solution in terms of $\{p_k(z)\}$.

LINEAR DEPENDENCE - THE WRONSKIAN.

A criterion to determine if a set of functions $\{\psi_i(z)\}$ are linearly dependent is the following, due to Wronski (1821):
suppose \( \gamma_1, \gamma_2, \ldots, \gamma_n \) are analytic and regular throughout the domain \( D \) of the complex \( z \)-plane.

If,
\[
\sum_{i=1}^{n} C_i \gamma_i(z) = 0, \text{ in } D,
\]
\[
\Rightarrow \sum_{i=1}^{n} C_i \gamma_i^{(l)}(z) = 0, \quad l=1,2,\ldots
\]

The system of homogeneous linear algebraic equations, \( \sum_{i=1}^{n} C_i \gamma_i^{(l)}(z) = 0, \quad l=0,1,\ldots,n-1 \)
will have a non trivial solution for \( \{C_i\} \)
if and only if,
\[
\Delta = \det \begin{vmatrix} \gamma_1(z) & \gamma_2(z) & \ldots & \gamma_n(z) \\ \gamma_1'(z) & \gamma_2'(z) & \ldots & \gamma_n'(z) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_1^{(n-1)}(z) & \gamma_2^{(n-1)}(z) & \ldots & \gamma_n^{(n-1)}(z) \end{vmatrix} = 0
\]

\( \Delta \) is called the WRONSKIAN.

Therefore, the set \( \{\gamma_i(z)\} \) is linearly dependent, if and only if \( \Delta = 0 \).

Notice the importance of the condition of analyticity of \( \gamma_i(z) \). Let us see this with the example:
\[\gamma_1 = \begin{cases} x^2, & x \geq 0 \\ 0, & x \leq 0 \end{cases}\]
\[\gamma_2 = \begin{cases} 0, & x \geq 0 \\ x^2, & x \leq 0 \end{cases}\]
Obviously, $\gamma_1$ and $\gamma_2$ are not analytic.

\[
\Delta = \begin{vmatrix} x^2 & 0 \\ 2x & 0 \end{vmatrix} = 0, \quad x \geq 0
\]

\[
\Delta = \begin{vmatrix} 0 & x^2 \\ 0 & 2x \end{vmatrix} = 0, \quad x \leq 0
\]

However, this two continuous (and differentiable) functions are not linearly dependent.

Going back to our problem (2nd order diff. equations),

\[
\gamma'' + p(z) \frac{d\gamma}{dz} + q(z) \gamma = 0
\]

Suppose two solutions $\gamma_1, \gamma_2$.

\[
\Delta(\gamma_1, \gamma_2) = \gamma_1 \gamma_2' - \gamma_1' \gamma_2
\]

\[
\frac{d\Delta}{dz} = \gamma_1 \gamma_2'' - \gamma_1'' \gamma_2 = \gamma_1(-p \gamma_2' - q \gamma_2) - \gamma_2(-p \gamma_1' - q \gamma_1)
\]

\[
= -p (\gamma_1 \gamma_2' - \gamma_1' \gamma_2) = -p \Delta
\]

\[
\Rightarrow \Delta(z) = e^{\frac{-\int p(z) dz}{2}} \Delta(z_0)
\]

**ABEL IDENTITY (**)**

\[
\Delta(z) \text{ depends on the functions } \gamma_1, \gamma_2, \text{ but the ratio of } \Delta \text{ in two points,}
\]

\[
\frac{\Delta(z)}{\Delta(z_0)} = e^{\int_{z_0}^z p(z') dz'}
\]

This is independent of $\gamma_1, \gamma_2$.

(**) *Abel identity is also valid for equations of any order n.*
Therefore, if $\Delta$ vanishes in a point $z_0$ of the domain $D$, it also vanishes in all the domain.

The Wronskian can be used to find a second solution of a 2nd order differential, homogeneous equation, if linearly independent of a first solution known:

Let $\gamma_1(z)$ be a known solution of the equation. Suppose $\gamma_2(z)$ is a second solution, linearly independent of $\gamma_1(z)$ and whose functional form is unknown.

$$\Delta(z) = \gamma_1(z)^2 - \gamma_2(z)^2 = \gamma_1^2 \frac{d}{dz} \left( \frac{\gamma_2}{\gamma_1} \right)$$

$$\Rightarrow \frac{\gamma_2}{\gamma_1} = \int_{z_0}^{z} \frac{\Delta(z''')}{\gamma_1^2(z'')} \, dz''$$

By means of the differential equation we proved,

$$\Delta(z''') = \Delta(z_0) \, e^{-\int_{z_0}^{z} p(z') \, dz'}$$

$$\gamma_2(z) = \gamma_1(z) \int_{z_0}^{z} \frac{\Delta(z_0) \, e^{-\int_{z_0}^{z} p(z') \, dz'}}{\gamma_1^2(z'')} \, dz''$$

$$\gamma_2(z) = C \gamma_1(z) \int_{z_0}^{z} \frac{e^{-\int_{z_0}^{z} p(z') \, dz'}}{\gamma_1^2(z'')} \, dz''$$

This expression shows explicitly the linear independence.
between \( \gamma_1 \) and \( \gamma_2 \). \( \gamma_2(z) \) can be substituted in the dif. eq. and we see that it is, in fact, a solution. (See Morse & Feshback, VI, p525).

The same result could have been obtained by building an independent function \( \gamma_2 = \gamma_1 \chi(z) \), and by imposing that \( \gamma_2 \) be a solution we find a differential eq. (of 1st degree) for \( \chi(z) \) (Method of VARIATION OF PARAMETERS).

**ONE-DIMENSIONAL HELMHOLTZ EQUATION**

\[
\frac{d^2 \gamma}{dz^2} + k^2 \gamma = 0
\]

\( \gamma_1(z) = \sin kz \) is a solution. Then,

\[
\gamma_2(z) = C_1 \gamma_1(z) \int_{z_0}^{z} \frac{e^{-\int_{z_0}^{z''} p(z')dz'}}{\gamma_1^2(z'')} \, dz''
\]

\[
= C \sin kz \int_{z_0}^{z} \frac{1}{\sin^2 kz''} \, dz''
\]

\( \gamma_2(z) = C \sin kz \left[ \frac{1}{kz} \right] \left[ \cot kz - \cot k z_0 \right] \)

choosing the constants.

\( \gamma_2(z) = \cos kz \)
METHOD OF VARIATION OF PARAMETERS

Suppose we have a second order dif. eq. inhomogeneous: \( LY = F \). If we know two solutions (linearly independent) of the homogeneous equation \( LY = 0 \), then we can find the general solution of the inhomogeneous equation:

\[
\gamma(z) = \gamma_0(z) + \gamma_1(z) + B \gamma_2(z)
\]

\( \gamma_1, \gamma_2 \)

are two linearly independent solutions of the homogeneous equation.

Another way of solving the equation, rather than variation of parameters is that already mentioned in the last lecture:

\[
\gamma_0(z) = \gamma_1(z) \chi(z)
\]

\( \begin{align*}
L \gamma_1(z) &= 0 \\
L \gamma_0(z) &= F(z)
\end{align*} \)  

(Morse & Feshbach) \( \text{(V. I, pp. 529-530)} \)

Suppose,

\[
\gamma(z) = A(z) \gamma_1(z) + B(z) \gamma_2(z)
\]

we introduce the constraint,

\[
A' \gamma_1 + B' \gamma_2 = 0
\]

then,

\[
\gamma' = A \gamma_1' + B \gamma_2'
\]
\[ y'' = A' y_1'' + B y_2'' + A' y_1' + B' y_2' \]

\[ \Rightarrow L(y) = A y_1'' + B y_2'' + A' y_1' + B' y_2' + p(A y_1 + B y_2) + q(A y_1 + B y_2) = F \]

Since \( y_1 \) and \( y_2 \) are solutions of the homogeneous equation, \( L y_1 = L y_2 = 0 \), and,

\[ L y = A' y_1' + B' y_2' = F \]

Thus, we have found to simultaneous dif. eq. for \( A' \) and \( B' \):

\[
\left\{ \begin{align*}
A' y_1 + B' y_2 &= 0 \\
A' y_1' + B' y_2' &= F
\end{align*} \right.
\]

Since the wronskian of \( y_1, y_2 \) \( (\Delta) \) is not null, this system has solution:

\[ A' = -\frac{F y_2}{\Delta}, \quad B' = \frac{F y_1}{\Delta} \]

\[
\left\{ \begin{align*}
A(z) &= C_1 - \int_{z_0}^{z} \frac{F(z') y_2(z')}{\Delta(z')} \, dz' \\
B(z) &= C_2 + \int_{z_0}^{z} \frac{F(z') y_1(z')}{\Delta(z')} \, dz'
\end{align*} \right.
\]

Therefore, the solution of the inhomogeneous equation is:
\[ \gamma(z) = -\gamma_i(z) \int_{z_0}^{z} \frac{F(z')\gamma_i(z')}{\Delta(z')} \, dz' + \gamma_2(z) \int_{z_0}^{z} \frac{F(z')\gamma_i(z')}{\Delta(z')} \, dz' \]

\[ \gamma(z) = \gamma_i(z) + a\gamma_i(z) + a_{2}\gamma_2(z) \]

**Example.**

**Helmholtz equation:** \[ \frac{d^2\gamma}{dz^2} + k^2\gamma = F(z) \]

\[ \gamma_1 = \cos kz \quad \gamma_2 = \sin kz \]

\[ \Delta = \begin{vmatrix} \cos kz & \sin kz \\ \frac{k\sin kz}{k} & \cos kz \end{vmatrix} = k \]

\[ \gamma(z) = -\frac{\cos kz}{k} \int_{z_0}^{z} F(z') \sin kz' \, dz' + \frac{\sin kz}{k} \int_{z_0}^{z} F(z') \cos kz' \, dz' \]

\[ + a_1 \cos kz + a_2 \sin kz \]

\[ \gamma(z) = -\frac{1}{k} \int_{z_0}^{z} F(z') \sin(kz - z') \, dz' + a_1 \cos kz + a_2 \sin kz \]

Notice that \( \frac{1}{k} \sin(kz - z') \) is the Green function for this equation.

**Fundamental Set of Solutions.**

A fundamental set of solutions of an \( n \)th order diff. eq. is a set of \( n \) linearly independent functions which are solution of the
homogeneous equation, and with the boundary conditions: \( \gamma_1(0) = 1 \quad \gamma_1'(0) = 0 \)
\( \gamma_2(0) = 0 \quad \gamma_2'(0) = 1 \)

Ince, pp 119-121
Copson, chap. \( \S \) 10.11

CLASSIFICATION OF SINGULARITIES.
SERIES SOLUTION.

\[ \gamma'' + p \gamma' + q = 0 \]

if \( z_0 \) is an ordinary point
(p and q analytic at \( z_0 \)).

\[ \gamma(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad (\gamma \text{ is analytic}) \]

If \( z_0 \) is not an ordinary point,
we guess that the solution would be of the form:

\[ \gamma(z) = (z-z_0)^{\alpha} \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad \alpha \text{ real} \]
this kind of solution is called a REGULAR INTEGRAL.

If there is at least one solution of the form of a regular integral, the point $z_0$ is called REGULAR SINGULAR point. But if there is no solution of the form of a regular integral, $z_0$ is called irregular singular point.

$$\gamma(z) = (z-z_0)^{\infty} \sum_{n=0}^\infty a_n(z-z_0)^n$$

$z_0$ can be a pole, or an algebraic branch point of $\gamma(z)$.

**THEOREM**

$z_0$ is a regular singular point of the dif. eq. if $z_0$ is not an ordinary point, and

$$p(z) = \frac{1}{z-z_0} P(z), \quad q(z) = \frac{1}{(z-z_0)^2} Q(z)$$

where $P(z), Q(z)$ are analytic and regular in a neighborhood of $z_0$.

Lecture 26, Dec 9/85

**Final exam:** Friday, 12/20, 12:30 p.m. room 106.

**SOLUTION NEAR AN ORDINARY POINT.**

$z = z_0$ ordinary point

$$p(z) = p_0 + p_1 \xi + p_2 \xi^2 + \ldots \quad \xi = z - z_0$$
\( \eta(z) = \eta_0 + \eta_1 z + \eta_2 z^2 + \ldots \quad |z| < R \)

\( \gamma(z) = \gamma_0 + \gamma_1 z + \gamma_2 z^2 + \ldots \)

\( \gamma'(z) = \gamma_1 + 2\gamma_2 z + 3\gamma_3 z^2 + \ldots \)

\( \gamma''(z) = 2\gamma_2 + 6\gamma_3 z + 12\gamma_4 z^2 + \ldots \)

\( \gamma'' + p \gamma' + q \gamma = \left( 2\gamma_2 + 6\gamma_3 z + 12\gamma_4 z^2 + \ldots \right) + \left( \gamma_1 + 2\gamma_2 z + 3\gamma_3 z^2 + \ldots \right)(p_0 + p_1 z + p_2 z^2 + \ldots) + \left( \gamma_0 + \gamma_1 z + \gamma_2 z^2 + \ldots \right)(q_0 + q_1 z + q_2 z^2 + \ldots) = 0 \)

Equate coefficients of equal powers of \( z \), we obtain:

\( \frac{\eta_0}{\xi} : 2\eta_2 + \eta_1 p_0 + \eta_0 q_0 = 0 \)

\( \frac{\eta_1}{\xi} : 6\eta_3 + 2\eta_2 p_0 + \eta_1 p_1 + \eta_0 q_1 = 0 \)

\( \vdots \)

Given \( \eta_0, \eta_1 \), the first equation allows us to find \( \eta_2 \), then the second \( \eta_3 \), and so on.

\( \eta_0 = \gamma(z_0) \quad \eta_1 = \gamma'(z_0) \)

Therefore we have found the complete solution \( \gamma(z) = \sum_{i=0}^{\infty} \epsilon_i \xi^i \) which obeys the two initial conditions.

This series solution converges in a neighborhood of \( z_0 \), \( |z| < R \).
Solution near a singular regular point.

\[ p = \frac{1}{\xi^5} P(\xi) \quad q = \frac{1}{\xi^2} Q(\xi) \]

\[ P(\xi) = p_0 + p_1 \xi + p_2 \xi^2 + \ldots \]
\[ Q(\xi) = q_0 + q_1 \xi + q_2 \xi^2 + \ldots \]

where \( p_0, q_0, q_1 \) are not all zero.

\[ \psi = \xi^{\alpha} (a_0 + a_1 \xi + a_2 \xi^2 + \ldots) \]
\[ a_0 \neq 0, \quad \alpha = \text{const.} \]

\[ \lambda = \begin{cases} \text{positive integer : } \xi \text{ is an ordinary point} \\ \text{neg. integer : } \xi \text{ is a pole of } \psi \\ \text{otherwise : } \xi \text{ is a branch point of } \psi \end{cases} \]

\[ \psi' = \lambda a_0 \xi^{\alpha-1} + (\lambda + 1) a_1 \xi^\alpha + (\lambda + 2) a_2 \xi^{\alpha+1} + \ldots \]
\[ \psi'' = \lambda (\lambda - 1) a_0 \xi^{\alpha-2} + (\lambda + 1) \lambda a_1 \xi^{\alpha-1} + (\lambda + 2) (\lambda + 1) a_2 \xi^\alpha + \ldots \]

\[ \psi'' + \frac{1}{\xi^5} P(\xi) \psi' + \frac{1}{\xi^2} Q(\xi) \psi = 0 = \xi^{2 \alpha} \psi'' + \xi^\alpha \psi' + q(\xi) \]

\[ \Rightarrow (\lambda (\lambda - 1) a_0 \xi^{\alpha-2} + (\lambda + 1) \lambda a_1 \xi^{\alpha-1} + (\lambda + 2) (\lambda + 1) a_2 \xi^\alpha + \ldots) + (p_0 + p_1 \xi + p_2 \xi^2 + \ldots) (\lambda a_0 \xi^{\alpha-1} + (\lambda + 1) a_1 \xi^\alpha + \ldots) + (q_0 + q_1 \xi + q_2 \xi^2 + \ldots) (\lambda a_0 \xi^\alpha + (\lambda + 1) a_1 \xi^{\alpha+1} + \ldots) = 0 \]
\[ \xi_0 : \quad \alpha (\alpha - 1) \xi_0 + \alpha \xi_0 \rho_0 + \xi_0 Q_0 = 0 \]

\[ \xi_{\alpha+1} : \quad \alpha (\alpha + 1) \xi_1 + (\alpha + 1) \xi_1 \rho_0 + \alpha \xi_0 \rho_1 + \xi_1 Q_0 + \xi_0 Q_1 = 0 \]

\[ \xi_0 \neq 0 \quad \Rightarrow \quad \boxed{\xi^2 + (\rho_0 - 1) \xi + Q_0 = 0} \quad \equiv F(\alpha) \quad (1) \]

\[ \exists \xi F(\alpha+1) = -\xi_0 (\alpha \xi_1 + Q_1) \quad (2) \]

\[ \xi_{\alpha+n} : \quad \xi_0 F(\alpha+n) = -\sum_{k=0}^{n-1} \alpha k \left( \alpha + k \right) P_{n-k} + Q_{n-k} \quad (3) \]

The equation \( F(\alpha) = 0 \) is called the indicial equation, and it allows us to determine \( \alpha \); but from the two values of \( \alpha \) found from the indicial equation we must decide which one gives us a solution.

Let \( \alpha_1 \) be one root of the indicial equation; equation (3) can be used to find \( \alpha_n \), if \( F(\alpha+n) \neq 0 \), this means that \( \alpha_1+n \) must not be another root of \( F(\alpha) \).

\[ \alpha_{1+n} \neq \alpha_2 \]

Therefore, if the two roots of the indicial equation do not differ by an integer, the recurrence relations lead us to two linearly independent solutions.
THEOREM.

If $P(z) = \sum_{n=0}^{\infty} p_n z^n$, $Q(z) = \sum_{n=0}^{\infty} q_n z^n$ are analytic and regular in $|z| < R$, then the series $\sum_{n=0}^{\infty} a_n z^n$ associated with the root $\alpha$ of the indicial equation, converges absolutely and uniformly throughout $|z| < R$.

Proof: Copson, p. 238-240.

If $z = z_0$ is a regular singular point, and the two roots of the indicial equation do not differ by an integer, both solutions are regular integrals.

\[
\gamma = \frac{\nu}{\alpha} \sum_{k=0}^{\infty} a_k \frac{z^k}{k!} \quad (\alpha \neq 0, \frac{\nu}{\alpha} \neq z - z_0)
\]

\[
\begin{cases}
\forall n \quad F(\alpha + n) = - \sum_{k=0}^{n-1} a_k \left( (\alpha+k)P_{n-k} + Q_{n-k} \right), \quad n = 1, 2, \ldots \\
F(\alpha) = \alpha^2 + (P_0 - 1)\alpha + Q_0 \\
F(\alpha_1) = F(\alpha_2) = 0
\end{cases}
\]

\]
\( \alpha_1 = \alpha_2 \)

we obtain just one solution \((\alpha_2 = \text{const} \times \alpha_0)\)
(the other one is a multiple of the first).

\( L_2 - L_1 = n \quad \) a positive integer

\[ F(\alpha_1) = 0 \]
\[ F(\alpha_1 + n) = 0 \]

\[ \Rightarrow \, \alpha_m \, , \, m \geq n \quad \text{can not be determined.} \]

In general, the method breaks down in the construction of one of the solutions.

However, the solution for \( \alpha_2 \), does not present any problem, since

\[ \alpha_2 + m > \alpha_2 > \alpha_1 \quad \forall \, m = 1, 2, 3, \ldots \]

And therefore \( F(\alpha_2 + m) \) is never zero.

Then, the method described, always gives us at least one solution.

A second solution can be found, as has already been mentioned:

\[ \gamma_2(z) = \gamma_1(z) \int_{z_0}^{z} \frac{1}{\gamma_1(w)} e^{-\frac{1}{2} \delta p(w) dw} \]
\( \gamma_0(z) = g_0 \gamma_0(z) \log \frac{z}{\alpha} + g_0 \sum_{n=1}^{\infty} \frac{b_n}{\alpha^n} \)

if \( n \neq 0 \),

\[ \gamma_n = g_n \gamma_n(z) \log \frac{z}{\alpha} + g_n \sum_{r=0}^{\infty} c_n \alpha^n \]

where \( \alpha \) is the root of the indicial equation which gave us the first solution \( \gamma_1(z) \).

Lecture 27, Dec 11.

Correction: if \( \alpha \) is not an integer:

\[ \alpha = \begin{cases} \text{rational number, } \alpha_0 \text{ is an} \\ \text{algebraic branch point} \\ \text{of the solution } \gamma(z) \\ \text{irrational, } \alpha_0 \text{ is a logarithmic} \\ \text{branch point.} \end{cases} \]

ASYMPTOTIC SOLUTIONS.

If \( |z| \to \infty \), an asymptotic solution may be easier to obtain, then the general solution.

\[ z = \frac{1}{w} \quad \gamma(z) = \overline{\Phi}(w) \]

\[ \begin{cases} \gamma' = -W^2 \overline{\Phi} \\ \gamma'' = 2W^3 \overline{\Phi} + W^4 \overline{\Phi}'' \end{cases} \]

where \( \overline{\Phi} = \frac{d \Phi}{dw} \)
\[ \ddot{\Phi}(w) + P(w)\dot{\Phi}(w) + Q(w)\Phi(w) = 0 \]

where:

\[ P(w) = \frac{2}{W} - \left(\frac{1}{w^2}\right) p\left(\frac{1}{w}\right) \]

\[ Q(w) = \frac{1}{W^2} q\left(\frac{1}{w}\right) \]

Example:

\[ p = 0 \]
\[ q = k^2 = \text{const} \]

\[ \gamma'' + \gamma'k^2 = 0 \]

\[ P(w) = \frac{2}{W} \]

\[ Q(w) = \frac{k^2}{W^2} \]

\[ \Rightarrow \text{the point at infinity is an irregular singularity.} \]

**Remarks on the Solution of Homogeneous Diff. Eq.**

\[ \psi''(z) + p(z)\psi'(z) + q(z)\psi(z) = 0 \]

An interesting case is when the number of singularities is less or equal to 3. Bilinear mapping can be used to obtain a new equation with singularities at 0, 1 and \( \infty \).

\[ z = \frac{aw + b}{cw + d} \quad (ad - bc) \neq 0 \]
It can be proved that such mapping doesn't change the roots of the indicial equation. For more than 3 singularities, there is no mapping which transforms the singularities into selected points, keeping the same roots of the indicial and without introducing new singularities.

For 1 to 3 singularities, there are 9 different cases depending on how many regular and irregular singularities are present. (table pp. 664-666, Morse & Feshbach).

**EQUATIONS WITH 3 REGULAR SINGULARITIES.**

(Riemann, 1857, Peperitz, 1857).

Let \( z_1, z_2, z_3 \) be the 3 singularities, and \( \lambda, \lambda' \) the exponents for \( z_1 \), \( \beta, \beta' \) those for \( z_2 \), and \( \gamma, \gamma' \) for \( z_3 \).

This 9 numbers completely specify the solution, but they are not independent:

\[ \lambda + \lambda' + \beta + \beta' + \gamma + \gamma' = 1 \]
therefore, the equation can be written as:

\[ q = \gamma(z) + \left( \sum_{i=1}^{3} \frac{1 - \alpha_i \beta_i}{z - z_i} \right) \gamma(z) + \sum_{i=1}^{3} \frac{(z_i - z_2)(z_i - z_3) \alpha_i \beta_i}{z - z_i} \frac{d \gamma}{dz} \]

where: \( \alpha_1 \equiv \alpha \), \( \alpha_2 \equiv \beta \), \( \alpha_3 \equiv \gamma \)

or also in the form obtained cyclically permuting

the indices \( 1, 2, 3 \) and \( \alpha, \beta, \gamma \).

This equation is called generalized hypergeometric equation.

We will label the solution as:

\[ \gamma(z) = P \left\{ \frac{z_1}{\alpha}, \frac{z_2}{\beta}, \frac{z_3}{\gamma} \right\} \]

Introducing a bilinear mapping,

\( (z, z_1, z_2, z_3) \rightarrow (w, w_1, w_2, w_3) \).

we obtain the solution in terms of \( w \), but with the same exponents.

\[ \gamma(z) \rightarrow P \left\{ \frac{w_1}{\alpha}, \frac{w_2}{\beta}, \frac{w_3}{\gamma} \right\} \]
Let us choose \( w_1 = 0 \), \( w_2 = 1 \), \( w_3 \to \infty \).

We already showed that the bilinear mapping must be of the form:

\[
W = \frac{z_2 - z_3}{z_2 - z_1} \frac{z - z_1}{z - z_3}
\]

and, thus, we only have 6 parameters now:

\[
\gamma(w) = P \left\{ \begin{array}{ccc}
\alpha & 1 & \infty \\
\alpha' & \beta' & \\
\alpha'' & 
\end{array} \right. W
\]

The equation can be simplified by:

\[
\gamma(z) = z^\alpha(z-1)^\beta \phi(z) \quad (z \text{ is now the new variable } w)
\]

where we have separated the exponents of the singularities at \( z = 0 \) and \( z = 1 \).

\[
\Rightarrow \phi(z) = P \left\{ \begin{array}{ccc}
0 & 1 & \infty \\
0 & 0 & \alpha + \beta + \gamma \\
\alpha - \alpha' & \beta' & \alpha + \beta + \gamma'
\end{array} \right. z
\]

\[
z(z-1)\phi''(z) + \left[ (1+\alpha+b)z - c \right] \phi'(z) + \phi(z) = 0
\]

where:
\[
\alpha = \alpha + \beta + \gamma \\
b = \alpha + \beta + \gamma' \\
c = 1 + \alpha - \alpha'
\]

The solution is then:

\[
\phi(z) = P \left\{ \begin{array}{ccc}
0 & 0 & \infty \\
1 - c & \alpha' & \\
\frac{z}{b}
\end{array} \right. = F(\alpha, b; c; z)
\]
By series method $F$ can be found:

$$F(a, b, c; z) = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{2! \; c(c+1)} z^2$$

$$+ \frac{a(a+1)(a+2)b(b+1)(b+2)}{3! \; c(c+1)(c+2)} z^3 + \ldots$$

Near $z=0$,

$$\phi(z) = A_0 \; F(a, b, c; z) + B_0 \; z^{1-c} \; F(a+1-c, b+1-c, z-c; z)$$

$$F(-n, 1, 1; -x) = 1 + nx + \frac{n(n+1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots$$

$$= (1-x)^n$$

$$\log(1+x) = x \; F(1, 1, 2; -x)$$

$$\sin^{-1} x = x \; F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; -x^2\right)$$

$$\tan^{-1} x = x \; F\left(\frac{1}{2}, 1, \frac{3}{2}; -x^2\right)$$

Examples:
1. LEGENDRE EQUATION.

\[(1-z^2)\psi'' - 2z \psi' + r(r+1)\psi = 0\]

It is easily shown that there are regular singular points at \(1, -1, \infty\), and

\[\gamma(z) = \mathcal{P} \begin{pmatrix} -1 & \infty & 1 & z \\ 0 & r+1 & 0 & 0 \\ 0 & -r & 0 & 0 \end{pmatrix}\]

2. ASSOCIATE LEGENDRE EQUATION.

\[(1-z^2)\psi'' - 2z \psi' + \left[\frac{r(r+1)}{1-z^2} - \frac{\mu^2}{1-z^2}\right]\psi = 0\]

Regular sing. points: \(1, -1, \infty\)

\[\gamma(z) = \mathcal{P} \begin{pmatrix} -1 & \infty & 1 & z \\ -\mu/2 & r-1 & -\mu/2 & 0 \\ +\mu/2 & -r & +\mu/2 \end{pmatrix}\]

3. BESSEL'S EQUATION.

\[\psi'' - \frac{1}{z} \psi' + \left(1 - \frac{r^2}{z^2}\right)\psi = 0\]

\(z=0\) is a regular singularity

\(z \to \infty\) is an irregular singularity, which can be treated as the COALESCENCE of two regular singularities.
The solution can not be found as a hypergeometric function, but as a similar function called CONFLUENT HYPERGEOMETRIC SERIES.

ORTHOGONAL POLYNOMIALS.

For most orthogonal polynomials, from the recurrence relation, a 2nd order diff. equation can be found which reduces to a hypergeometric equation. Such polynomials are called JACOBI polynomials; examples are:

Legendre polynomials
Laguerre ""
Hermite ""
Tchebycheff ""
Kummer's 24 Solutions of the Hypergeometric Equation

\[ u_1 = \left( \frac{z-a}{z-b} \right)^{a} \left( \frac{z-c}{z-b} \right)^{c} F \left\{ a + \beta + \gamma, \ a + \beta' + \gamma'; \ 1 + a - a'; \ \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\} \]

\[ u_2 = \left( \frac{z-a}{z-b} \right)^{a} \left( \frac{z-c}{z-b} \right)^{c} F \left\{ a' + \beta + \gamma, \ a' + \beta' + \gamma'; \ 1 + a' - a; \ \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\} \]

\[ u_3 = \left( \frac{z-a}{z-b} \right)^{a} \left( \frac{z-c}{z-b} \right)^{c} F \left\{ a' + \beta + \gamma', \ a' + \beta' + \gamma'; \ 1 + a' - a; \ \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\} \]

\[ u_4 = \left( \frac{z-a}{z-b} \right)^{a} \left( \frac{z-c}{z-b} \right)^{c} F \left\{ a' + \beta + \gamma', \ a' + \beta' + \gamma'; \ 1 + a' - a; \ \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\} \]

\[ u_5 = \left( \frac{z-b}{z-c} \right)^{b} \left( \frac{z-a}{z-c} \right)^{a} F \left\{ \beta + \gamma + \alpha, \ \beta + \gamma' + \alpha; \ 1 + \beta - \beta'; \ \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\} \]

\[ u_6 = \left( \frac{z-b}{z-c} \right)^{b} \left( \frac{z-a}{z-c} \right)^{a} F \left\{ \beta + \gamma + \alpha, \ \beta + \gamma' + \alpha; \ 1 + \beta - \beta'; \ \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\} \]

\[ u_7 = \left( \frac{z-b}{z-c} \right)^{b} \left( \frac{z-a}{z-c} \right)^{a} F \left\{ \beta + \gamma + \alpha', \ \beta + \gamma' + \alpha'; \ 1 + \beta - \beta'; \ \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\} \]

\[ u_8 = \left( \frac{z-b}{z-c} \right)^{b} \left( \frac{z-a}{z-c} \right)^{a} F \left\{ \beta' + \gamma + \alpha, \ \beta' + \gamma' + \alpha; \ 1 + \beta' - \beta; \ \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\} \]

\[ u_{10} = \left( \frac{z-b}{z-a} \right)^{b} \left( \frac{z-a}{z-c} \right)^{a} F \left\{ \gamma + \alpha + \beta, \ \gamma + \alpha' + \beta; \ 1 + \gamma - \gamma'; \ \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\} \]

\[ u_{11} = \left( \frac{z-b}{z-a} \right)^{b} \left( \frac{z-a}{z-c} \right)^{a} F \left\{ \gamma + \alpha + \beta, \ \gamma + \alpha' + \beta; \ 1 + \gamma - \gamma'; \ \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\} \]

\[ u_{12} = \left( \frac{z-b}{z-a} \right)^{b} \left( \frac{z-a}{z-c} \right)^{a} F \left\{ \gamma + \alpha + \beta', \ \gamma + \alpha' + \beta'; \ 1 + \gamma - \gamma'; \ \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\} \]

\[ u_{13} = \left( \frac{z-a}{z-c} \right)^{a} \left( \frac{z-b}{z-c} \right)^{b} F \left\{ \alpha + \gamma + \beta, \ a + \gamma' + \beta; \ 1 + a - a'; \ \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\} \]

\[ u_{14} = \left( \frac{z-a}{z-c} \right)^{a} \left( \frac{z-b}{z-c} \right)^{b} F \left\{ \alpha + \gamma + \beta, \ a + \gamma' + \beta; \ 1 + a - a'; \ \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\} \]

\[ u_{15} = \left( \frac{z-a}{z-c} \right)^{a} \left( \frac{z-b}{z-c} \right)^{b} F \left\{ \alpha' + \gamma + \beta, \ \alpha' + \gamma' + \beta; \ 1 + a' - a; \ \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\} \]

\[ u_{16} = \left( \frac{z-a}{z-c} \right)^{a} \left( \frac{z-b}{z-c} \right)^{b} F \left\{ \alpha' + \gamma + \beta, \ \alpha' + \gamma' + \beta; \ 1 + a' - a; \ \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\} \]

\[ u_{17} = \left( \frac{z-a}{z-a} \right)^{a} \left( \frac{z-b}{z-b} \right)^{b} F \left\{ \gamma + \beta + \alpha, \ \gamma + \beta' + \alpha; \ 1 + \gamma' - \gamma; \ \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\} \]

\[ u_{18} = \left( \frac{z-a}{z-a} \right)^{a} \left( \frac{z-b}{z-b} \right)^{b} F \left\{ \gamma + \beta + \alpha, \ \gamma + \beta' + \alpha; \ 1 + \gamma' - \gamma; \ \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\} \]

\[ u_{19} = \left( \frac{z-c}{z-b} \right)^{c} \left( \frac{z-a}{z-b} \right)^{a} F \left\{ \gamma + \beta + \alpha', \ \gamma + \beta' + \alpha'; \ 1 + \gamma' - \gamma; \ \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\} \]

\[ u_{20} = \left( \frac{z-c}{z-b} \right)^{c} \left( \frac{z-a}{z-b} \right)^{a} F \left\{ \gamma + \beta + \alpha', \ \gamma + \beta' + \alpha'; \ 1 + \gamma' - \gamma; \ \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\} \]

\[ u_{21} = \left( \frac{z-b}{z-a} \right)^{b} \left( \frac{z-c}{z-a} \right)^{c} F \left\{ \beta + a + \gamma, \ \beta + a' + \gamma; \ 1 + \beta - \beta'; \ \frac{(c-a)(z-b)}{(c-b)(z-a)} \right\} \]

\[ u_{22} = \left( \frac{z-b}{z-a} \right)^{b} \left( \frac{z-c}{z-a} \right)^{c} F \left\{ \beta + a + \gamma, \ \beta + a' + \gamma; \ 1 + \beta - \beta'; \ \frac{(c-a)(z-b)}{(c-b)(z-a)} \right\} \]

\[ u_{23} = \left( \frac{z-b}{z-a} \right)^{b} \left( \frac{z-c}{z-a} \right)^{c} F \left\{ \beta + a + \gamma', \ \beta + a' + \gamma'; \ 1 + \beta - \beta'; \ \frac{(c-a)(z-b)}{(c-b)(z-a)} \right\} \]

\[ u_{24} = \left( \frac{z-b}{z-a} \right)^{b} \left( \frac{z-c}{z-a} \right)^{c} F \left\{ \beta + a + \gamma', \ \beta + a' + \gamma'; \ 1 + \beta - \beta'; \ \frac{(c-a)(z-b)}{(c-b)(z-a)} \right\} \]